Section 2.3

Matrix Equations

Definition

The **product** of A with a vector x in $\mathbf{R}^n$ is the linear combination

$$Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \overset{\text{def}}{=} x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

this means the equality is a *definition*

these must be equal

The output is a vector in $\mathbf{R}^m$.

Example

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} =$$

Note that the number of columns of A has to equal the number of rows of x.

Matrix Equations

An example

Question

Let v_1, v_2, v_3 be vectors in $\mathbf{R}^3$. How can you write the vector equation

$$2v_1 + 3v_2 - 4v_3 = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}$$

in terms of matrix multiplication?

$$\boxed{A\vec{x} = \vec{b}}$$

$$\underbrace{\begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}}_{\vec{b}}$$

Matrix Equations

In general

Let $v_1, v_2, \ldots, v_n$, and b be vectors in $\mathbf{R}^m$. Consider the vector equation

vectors.

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b.$$

→ vectors

→ #s

It is equivalent to the **matrix equation**

$$Ax = b$$

where

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Conversely, if A is any $m \times n$ matrix, then

$$Ax = b \quad \text{is equivalent to the vector equation} \quad x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b$$

where $v_1, \ldots, v_n$ are the columns of A, and $x_1, \ldots, x_n$ are the entries of x.

$\boxed{\text{Note}}$ $\underbrace{\vec{v}_1, \ldots, \vec{v}_n}_{n \text{ vectors}}$

Each vector has m components (in $\mathbb{R}^m$)

Linear Systems, Vector Equations, Matrix Equations, ...

We now have *four* equivalent ways of writing (and thinking about) linear systems:

1. As a system of equations:
$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 - x_2 &= 5 \end{aligned}$$
2. As an augmented matrix:
$$\left(\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 5 \end{array}\right)$$
3. As a vector equation ($x_1 v_1 + \cdots + x_n v_n = b$):
$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$
4. As a matrix equation ($Ax = b$):
$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

In particular, *all four have the same solution set.*

We will move back and forth freely between these over and over again, for the rest of the semester. Get comfortable with them now!

Q: Do they intersect? How do they intersect?

Q: RREF, Pivots, free variable, soln sets

Q: Is $\vec{b} \in \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_n\}$. Are there $x_1, \dots, x_n$ s.t. $\vec{b}$ is a lin. comb.

Q: Later chapter. transformation. maps

**

Very Important Fact That Will Appear on Every Midterm and the Final

$Ax = b$ has a solution

$\iff$ there exist $x_1, \dots, x_n$ such that $A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b$

"if and only if"

$\iff$ there exist $x_1, \dots, x_n$ such that $x_1 v_1 + \cdots + x_n v_n = b$

$\iff$ b is a linear combination of $v_1, \dots, v_n$

$\iff$ b is in the span of the columns of A.

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ have a solution?

https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=2,-1,1&v2=1,0,-1&target=0,2,2&tlabel=b&range=5

Can you reframe this question in other ways? $\boxed{4}$ equivalent ways

① $\begin{cases} 2x_1 + x_2 = 0 \\ -x_1 = 2 \\ x_1 - x_2 = 2 \end{cases}$

② $\left(\begin{array}{cc|c} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 1 & -1 & 2 \end{array}\right) \xrightarrow{\text{Row Reduce}} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) \Rightarrow$ No soln. $\Rightarrow A\vec{x} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ has no soln.

③ $x_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$

$\boxed{\text{② is helpful finding the soln}}$

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ have a solution?

$\left(\begin{array}{cc|c} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 2 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{matrix} x_1 = 1 \\ x_2 = -1 \end{matrix}$

$\Rightarrow Ax = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ has soln $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\boxed{\text{check}}$ $\begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2-1 \\ -1+0 \\ 1+1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ ✓

When Solutions Always Exist

rows : # of eqns

Here are criteria for a linear system to *always* have a solution.

col. : # of unknowns

Theorem

Let A be an $m \times n$ (non-augmented) matrix. The following are equivalent:

1. $Ax = b$ has a solution *for all* b in $\mathbf{R}^m$.
2. The span of the columns of A is all of $\mathbf{R}^m$.
3. A has a pivot in each row.

recall that this means that for given A, either they're all true, or they're all false

Example where conditions ARE satisfied

https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=2,-1,1&v2=1,0,-1&v3=-1,2,2&range=5&capopt=matrix

Example where conditions are NOT satisfied

https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=2,-1,1&v2=1,0,-1&v3=1,-1,2&range=5&capopt=matrix

Q: What if we change 3 to

A has a pivot in each $\boxed{\text{col}}$

Is it still true? → $\boxed{\text{No}}$

$$\left(\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{array}\right)$$

$\begin{pmatrix} a \\ b \\ 1 \end{pmatrix}$ is not in span$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

Properties of the Matrix–Vector Product

Let c be a scalar, u, v be vectors, and A a matrix.

- $A(u + v) = Au + Av$ - - - (*)
- $A(cv) = cAv$ - - - - (**)

Consequence: If u and v are solutions to $Ax = 0$, then so is every vector in Span$\{u, v\}$. Why?

$A\vec{u} = \vec{0}$

$A\vec{v} = \vec{0}$

Consider an element in span$\{u, v\}$, $c_1\vec{u} + c_2\vec{v}$ where c_1, c_2 are scalars.

Now $A(c_1\vec{u} + c_2\vec{v}) = \underset{(*)}{Ac_1\vec{u} + Ac_2\vec{v}} = \underset{(**)}{c_1A\vec{u} + c_2A\vec{v}} = c_1\vec{0} + c_2\vec{0}$

$= \vec{0}$

Important

The set of solutions to $Ax = 0$ is a span.

↳ span$\{\vec{u}_1, \cdots, \vec{u}_m\}$ for some vectors.

Summary

- We have four equivalent ways of writing a system of linear equations:
  1. As a system of equations.
  2. As an augmented matrix.
  3. As a vector equation.
  4. As a matrix equation $Ax = b$.
- $Ax = b$ is consistent if and only if b is in the span of the columns of A. The latter condition is geometric: you can draw pictures of it.
- $Ax = b$ is consistent for all b in $\mathbf{R}^m$ if and only if the columns of A span $\mathbf{R}^m$.

⟺ pivot in each row.

Make sure you are comfortable reframing in all 4 ways.

Section 2.4

Solution Sets

Plan For Today

Today we will learn to describe and draw the solution set of an arbitrary system of linear equations $Ax = b$, using spans.

Recall: the **solution set** is the collection of all vectors x such that $Ax = b$ is true.

Homogeneous Systems

Definition
A system of linear equations of the form $Ax = 0$ is called **homogeneous.**

Definition
A system of linear equations of the form $Ax = b$ with $b \neq 0$ is called **inhomogeneous.**

A homogeneous system always has the solution $x = 0$. This is called the **trivial solution.** The nonzero solutions are called **nontrivial.**

Observation
$Ax = 0$ has a nontrivial solution
$\iff$ there is a free variable
$\iff$ A has a column with no pivot.

$A\vec{x} = \vec{0}$ must have at least 1 (trivial soln)

If it has more solns, the rest will be non-trivial

Does there exist $[0 \cdots 0 \mid k]$, $k \neq 0$

yes → No soln.

No → Are there free variables

yes → ∞ly many solns

No → exactly 1 soln.

Question
What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}?$$

Observation
Since the last column (everything to the right of the $=$) was zero to begin, it will always stay zero! So it's not really necessary to write augmented matrices in the homogeneous case.

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

The soln set is $\{\vec{0}\}$.

trivial soln. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\boxed{\text{Note}}$ No free variables exist

Homogeneous Systems
Example

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}?$$

$$\left(\begin{array}{cc|c} 1 & -3 & 0 \\ 2 & -6 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

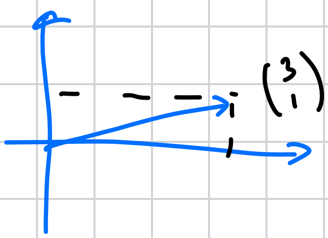
$1 \cdot x_1 + (-3)x_2 = 0$ $\qquad$ $x_1 = 3x_2$

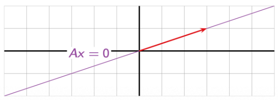
$x_2 = x_2$ (free) $\qquad$ $x_2 = x_2$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}}_{\text{span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}} \ (x_2 \text{ is real}).$$

parametric vector form.

Will be on test.





Note: *one* free variable means the solution set is a *line* in $\mathbf{R}^2$ ($2 = \#$ variables $= \#$ columns).

https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?lock=true&x=3,1&mat=1,-3:2,-6&range2=5

* span$\{\vec{v}\}$ where $\vec{v} \neq \vec{0}$ is a line

** Note it goes through the origin.

Question
What is the solution set of $Ax = 0$, where

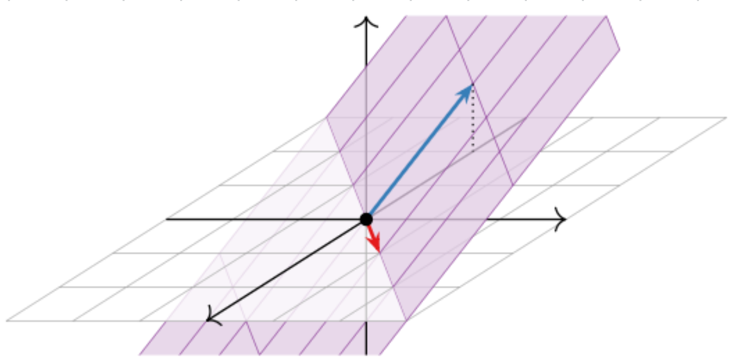
$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}?$$

Without row reduction, we notice that # pivots is $\boxed{\text{AT MOST}}$ 2. # cols = 3.
⇒ Free variable(s) exist(s)
⇒ There are non-trivial solns

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ -2 & 2 & -4 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 - x_2 + 2x_3 = 0 \\ x_2 = x_2 \\ x_3 = x_3 \end{cases} \quad \text{free} \rightsquigarrow \quad \begin{aligned} x_1 &= x_2 - 2x_3 \\ x_2 &= x_2 \\ x_3 &= x_3 \end{aligned}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}}_{\text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}\right\}.}, \quad x_2, x_3 \text{ are real #s}$$



https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?lock=true&x=0,0,0

Note: *two* free variables means the solution set is a *plane* in $\mathbf{R}^3$ ($3 = \#$ variables $= \#$ columns).

Question

What is the solution set of $Ax = 0$, where $A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}$

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ -2 & -3 & 4 & 5 & 0 \\ 2 & 4 & 0 & -2 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & -8 & -7 & 0 \\ 0 & 1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

If we stop here, how many vectors will span the soln set?

$\boxed{2}$ as there are 2 free variables

$$\begin{cases} x_1 \quad -8x_3 - 7x_4 = 0 \\ x_2 \quad +4x_3 + 3x_4 = 0 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases}$$

free

$$\begin{aligned} x_1 &= 8x_3 + 7x_4 \\ x_2 &= -4x_3 - 3x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

x_3, x_4 are real #s

$$\operatorname{span}\left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Note: *two* free variables means the solution set is a *plane* in $\mathbf{R}^4$ ($4 = \#$ variables $= \#$ columns).

Parametric Vector Form

Homogeneous systems

Let A be an $m \times n$ matrix. Suppose that the free variables in the homogeneous equation $Ax = 0$ are, for example, x_3, x_6, and x_8.

1. Find the reduced row echelon form of A.
2. Write the parametric form of the solution set, including the redundant equations $x_3 = x_3$, $x_6 = x_6$, and $x_8 = x_8$. Put equations for all of the x_i in order.
3. Make a single vector equation from these equations by putting x_3, x_6, and x_8 as coefficients of vectors v_3, v_6, and v_8, respectively.

The solutions to $Ax = 0$ will then be expressed in the form

$$x = x_3 v_3 + x_6 v_6 + x_8 v_8$$

for some vectors v_3, v_6, v_8 in $\mathbf{R}^n$, and any scalars x_3, x_6, x_8.

In this case, the solution set to $Ax = 0$ is

$$\operatorname{Span}\{v_3, v_6, v_8\}.$$

You should be very comfortable with this algo.

Inhomogeneous Systems
Example

Question
What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -3 \\ -6 \end{pmatrix}?$$

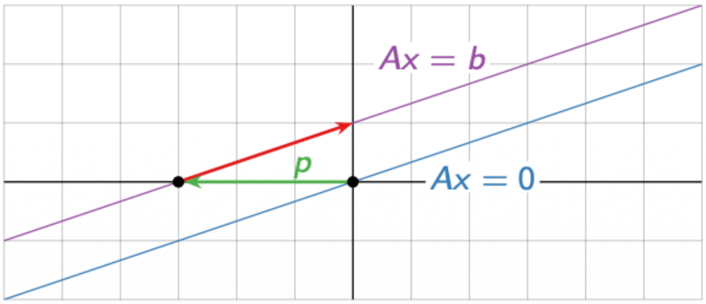
https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?x=-3,0&mat=1,-3:2,-6&lock=true&closed=true

$$\left(\begin{array}{cc|c} 1 & -3 & -3 \\ 2 & -6 & -6 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -3 & -3 \\ 0 & \underset{\text{free}}{0} & 0 \end{array}\right).$$

$$\begin{cases} x_1 - 3x_2 = -3 \\ \quad\ \ x_2 = x_2 \end{cases} \to \begin{matrix} x_1 = -3 + 3x_2 \\ x_2 = x_2 \end{matrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} -3 \\ 0 \end{pmatrix}}_{\vec{p}} + x_2 \underbrace{\begin{pmatrix} 3 \\ 1 \end{pmatrix}}_{\text{soln to homogeneous system } A\vec{x} = \vec{0}} \quad x_2 \text{ is real \#.}$$

Note
* The soln is not a span now
** It does not go through the origin.
*** Soln set shifted by $\vec{p}$



But what is $\vec{p}$?

$$A\vec{p} = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}\begin{pmatrix} -3 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \end{pmatrix}$$

$\vec{p}$ satisfies $A\vec{x} = \vec{b}$!!!

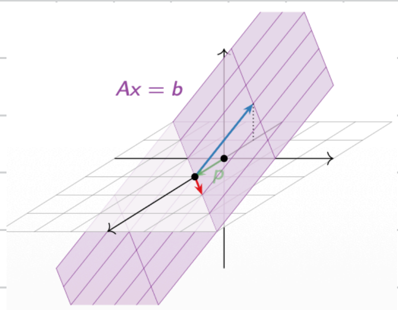
Inhomogeneous system has soln of the form.

$$\underset{\text{particular soln}}{\vec{p}} + \underset{\text{homogeneous soln.}}{\vec{h}}$$

Inhomogeneous Systems
Example

Question
What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}?$$



Homogeneous vs. Inhomogeneous Systems

Key Observation

The set of solutions to $Ax = b$, if it is nonempty, is obtained by taking one **specific** or **particular solution** p to $Ax = b$, and adding all solutions to $Ax = 0$.

Very Important

Let A be an $m \times n$ matrix. There are now two *completely different* things you know how to describe using spans:

- The **solution set:** for fixed b, this is all x such that $Ax = b$.
  - This is a span if $b = 0$, or a translate of a span in general (if it's consistent).
  - Lives in $\mathbf{R}^n$.
  - Computed by finding the parametric vector form.
- The **span of the columns:** this is all b such that $Ax = b$ is consistent.
  - This is the span of the columns of A.
  - Lives in $\mathbf{R}^m$.

Summary

- The solution set to a **homogeneous** system $Ax = 0$ is a span. It always contains the **trivial solution** $x = 0$.
- The solution set to a **nonhomogeneous** system $Ax = b$ is either empty, or it is a translate of a span: namely, it is a translate of the solution set of $Ax = 0$.
- The solution set to $Ax = b$ can be expressed as a translate of a span by computing the **parametric vector form** of the solution.
- The solution set to $Ax = b$ and the span of the columns of A (from the previous lecture) are two completely different things, and you have to understand them separately.