Section 2.6

Subspaces

https://strawpoll.com/

Motivation

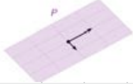
Today we will discuss **subspaces** of $\mathbf{R}^n$.

Examples

Example
A line L through the origin is a subspace: L contains zero and is easily seen to be closed under addition and scalar multiplication.

Example

A plane P through the origin is a subspace: P contains zero; the sum of two vectors in P is also in P; and any scalar multiple of a vector in P is also in P.



Example
All of $\mathbf{R}^n$: this contains 0, and is closed under addition and scalar multiplication.

Example
The subset $\{0\}$: this subspace contains only one vector.

Definition of Subspace

Definition
A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V. "not empty"
2. If u and v are in V, then $u + v$ is also in V. "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V. "closed under × scalars"

A subspace is a span of some set of vectors in it.

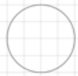
Subsets and Subspaces

They aren't the same thing

A **subset** of $\mathbf{R}^n$ is any collection of vectors in $\mathbf{R}^n$ whatsoever. For example, the unit circle

$$C = \{(x, y) \text{ in } \mathbf{R}^2 \mid x^2 + y^2 = 1\}$$

is a subset of $\mathbf{R}^2$, but it is not a subspace.



Subset: *yes*
Subspace: *no*

① $\begin{bmatrix}0\\0\end{bmatrix}$ in C? no

② $\begin{bmatrix}0\\1\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$ in C? No

③ $3\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\3\end{bmatrix}$ No.

Ex. Some vectors on C are

$\begin{bmatrix}0\\1\end{bmatrix}$ ✓, $\begin{bmatrix}1\\0\end{bmatrix}$ ✓, $\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$ ✓, $\begin{bmatrix}2\sqrt{2}/3\\1/3\end{bmatrix}$ ✓, $\begin{bmatrix}0\\0\end{bmatrix}$ ✗, $\begin{bmatrix}-1\\1\end{bmatrix}$ ✗, $\begin{bmatrix}1\\1\end{bmatrix}$ ✗

$0^2+1^2=1$ ✓ $1^2+0^2=1$ ✓ $\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2=\frac{1}{2}+\frac{1}{2}=1$ ✓ $\left(\frac{2\sqrt{2}}{3}\right)^2+\left(\frac{1}{3}\right)^2=\frac{4*2}{9}+\frac{1}{9}=\frac{9}{9}=1$ ✓

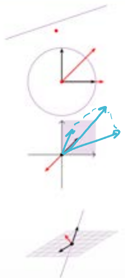
Non-Examples

Non-Example
A line L (or any other set) that doesn't contain the origin is not a subspace. Fails: 1.

Non-Example
A circle C is not a subspace. Fails: 1,2,3. Think: a circle isn't a "linear space."

Non-Example
The first quadrant in $\mathbf{R}^2$ is not a subspace. Fails: 3 only.

Non-Example
A line union a plane in $\mathbf{R}^3$ is not a subspace. Fails: 2 only.



Subspaces are Spans, and Spans are Subspaces

Theorem
Any Span$\{v_1, v_2, \ldots, v_p\}$ is a subspace. ✓

Definition
If $V = \text{Span}\{v_1, v_2, \ldots, v_p\}$, we say that V is the subspace **generated by** or **spanned by** the vectors $v_1, v_2, \ldots, v_p$. We call $\{v_1, v_2, \ldots, v_p\}$ a **spanning set** for V.

!!! Every subspace is a span, and every span is a subspace.

Poll

Poll
Which of the following are subspaces?
A. The empty set {}.
B. The solution set to a homogeneous system of linear equations.
C. The solution set to an inhomogeneous system of linear equations.
D. The set of all vectors in $\mathbf{R}^n$ with rational (fraction) coordinates.
For the ones which are not subspaces, which property(ies) do they not satisfy?

Example: check from the definitions that Span{v1,v2} is a subspace where v1=[1;0;0] and v2=[0;1;0].

① $\begin{bmatrix}0\\0\\0\end{bmatrix}$ in V?

② v_1, v_2 in V then v_1+v_2 also in V

③ v in V, c any real number then $c*v$ in V.

$V=\text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ is a subspace

① $\begin{bmatrix}0\\0\\0\end{bmatrix}=0\begin{bmatrix}1\\0\\0\end{bmatrix}+0\begin{bmatrix}0\\1\\0\end{bmatrix}$ is in $\text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

so $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is in V. ✓

② $v_1=a\begin{bmatrix}1\\0\\0\end{bmatrix}+b\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}a\\b\\0\end{bmatrix}$ Is v_1+v_2 in $\text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$?

$v_2=c\begin{bmatrix}1\\0\\0\end{bmatrix}+d\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}c\\d\\0\end{bmatrix}$ $v_1+v_2=\begin{bmatrix}a\\b\\0\end{bmatrix}+\begin{bmatrix}c\\d\\0\end{bmatrix}=\begin{bmatrix}a+c\\b+d\\0\end{bmatrix}=(a+c)\begin{bmatrix}1\\0\\0\end{bmatrix}+(b+d)\begin{bmatrix}0\\1\\0\end{bmatrix}$ ✓

③ $v=a\begin{bmatrix}1\\0\\0\end{bmatrix}+b\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}a\\b\\0\end{bmatrix}$

$c*v=c\begin{bmatrix}a\\b\\0\end{bmatrix}=\begin{bmatrix}ca\\cb\\0\end{bmatrix}=ca\begin{bmatrix}1\\0\\0\end{bmatrix}+cb\begin{bmatrix}0\\1\\0\end{bmatrix}$ ✓

You Try it!

Poll

Poll

Which of the following are subspaces?

A. The empty set {}. ✗

B. The solution set to a homogeneous system of linear equations. ✓

C. The solution set to an inhomogeneous system of linear equations. ✗

D. The set of all vectors in $\mathbf{R}^n$ with rational (fraction) coordinates. ✗

For the ones which are not subspaces, which property(ies) do they not satisfy?

collect all x's

$Ax = 0$

$Ax = b$

$x = \begin{bmatrix}1\\1\end{bmatrix}$ is a soln to $\begin{pmatrix}1&1\\1&1\end{pmatrix}x = \begin{pmatrix}2\\2\end{pmatrix}$

$$\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}$$

$y = \begin{pmatrix}0\\2\end{pmatrix}$ also a soln

$$\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{bmatrix}0\\2\end{bmatrix} = \begin{pmatrix}2\\2\end{pmatrix}$$

Check

$x+y = \begin{pmatrix}1\\3\end{pmatrix}$ is a soln?

$$\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{bmatrix}1\\3\end{bmatrix} = \begin{pmatrix}4\\4\end{pmatrix} \neq \begin{pmatrix}2\\2\end{pmatrix}.$$

Subspaces

Verification

Let $V = \left\{\begin{pmatrix}a\\b\end{pmatrix} \text{ in } \mathbf{R}^2 \mid ab = 0\right\}$. Let's check if V is a subspace or not.

$$\begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}5\\0\end{bmatrix}, \begin{bmatrix}0\\-6\end{bmatrix}, \ldots$$

① $\begin{bmatrix}0\\0\end{bmatrix}$ in V ✓

② $\begin{bmatrix}2\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$ is **Not** in V. ✗

③ $c\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}2c\\0\end{bmatrix}$ in V ✓

Definition of Subspace

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V. ✓ "not empty"
2. If u and v are in V, then $u + v$ is also in V. ✗ "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V. ✓ "closed under × scalars"

Column Space and Null Space

An $m \times n$ matrix A naturally gives rise to *two* subspaces.

Definition

- The **column space** of A is the subspace of $\mathbf{R}^m$ spanned by the columns of A. It is written Col A.
- The **null space** of A is the set of all solutions of the homogeneous equation $Ax = 0$:

$$\text{Nul } A = \{x \text{ in } \mathbf{R}^n \mid Ax = 0\}.$$

This is a subspace of $\mathbf{R}^n$.

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Definition of Subspace

Definition

A subspace of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. ✓ The zero vector is in V. "not empty"
2. ✓ If u and v are in V, then $u + v$ is also in V. "closed under addition"
3. ✓ If u is in V and c is in $\mathbf{R}$, then cu is in V. "closed under × scalars"

Example: Why is Col(A) always a subspace?

Span of any vectors is always a subspace

So $\text{Col } A = \text{span}\{\text{cols of } A\}$ is always a subspace of $\mathbb{R}^m$

where $m = \#$ rows of A.

Example: Show that Nul(A) always satisfies all the conditions for being a subspace, using the definitions.

how matrix mult. works.

① $A\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ so $\begin{bmatrix}0\\0\end{bmatrix}$ in Nul A

② $Ax = 0$ and $Ay = 0$ $\Rightarrow$ $A(x+y) = Ax + Ay = 0 + 0$ ✓

③ $Ax = 0$ and c in $\mathbb{R}$ $\Rightarrow$ $A(cx) = cAx = c \cdot 0 = 0$ ✓

Column Space and Null Space

Example

Let $A = \begin{pmatrix}1 & 1\\1 & 1\\1 & 1\end{pmatrix}$.

Example: Find Col(A) and Nul(A) where A=[1 1 ; 1 1 ; 1 1]

Column Space and Null Space

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This is a subspace of $\mathbf{R}^n$.

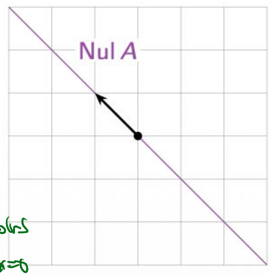
Col(A)

$\text{Col } A = \text{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ done.

$\text{Nul } A = \{\text{all solutions to } Ax = 0\}$

$\begin{bmatrix}1 & 1\\1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}*\\*\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$??

$\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}-2\\2\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}1/2\\-1/2\end{bmatrix}, \ldots$ are all in Nul A.



$\begin{bmatrix}1 & 1\\1 & 1\\1 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 1\\0 & 0\\0 & 0\end{bmatrix}$ (x_2 free)

$x_1 = -x_2$
$x_2 = x_2$ (free)

thus $x = x_2\begin{bmatrix}-1\\1\end{bmatrix}$ all sols to $Ax=0$

$\text{Nul } A = \text{span}\left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\}$

Example: with A as above, describe Col(A) and Nul(A) geometrically.

$\text{Col } A = \text{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ is a line in $\mathbb{R}^3$

$\text{Nul } A = \text{span}\left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\}$ is a line in $\mathbb{R}^2$

The Null Space is a Span

The column space of a matrix A is defined to be a span (of the columns).

The null space is defined to be the solution set to $Ax = 0$. It is a subspace, so it is a span.

Question

How to find vectors that span the null space?

punchline: row reduce & get parametric vector form of the soln to $Ax=0$ to get spanning set for subspace Nul A.

Example: Write the null space of the matrix A as a span.
A=[1 0 -1 2 ; 0 1 -2 -3]

(free: x_3, x_4)

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 3 \end{bmatrix}$$

Q: Write Nul(A) as a span.

$\text{Nul } A = \text{Span}\left\{ \begin{bmatrix} ? \end{bmatrix}, \begin{bmatrix} ? \end{bmatrix} \right\}$ (what goes here??)

general soln to $Ax=0$

$$\begin{cases} x_1 - x_3 + 2x_4 = 0 \\ x_2 - 2x_3 + 3x_4 = 0 \\ x_3 = x_3 \text{ (free)} \\ x_4 = x_4 \text{ (free)} \end{cases} \Rightarrow \begin{cases} x_1 = x_3 - 2x_4 \\ x_2 = 2x_3 - 3x_4 \\ x_3 = x_3 \text{ (free)} \\ x_4 = x_4 \text{ (free)} \end{cases} \Rightarrow x = \begin{bmatrix} x_3 - 2x_4 \\ 2x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

So x is a soln to $Ax=0$ exactly when

x in $\text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\} = \text{Nul}(A)$

Answer: Parametric vector form! We know that the solution set to $Ax = 0$ has a parametric form that looks like

$$x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

if, say, x_3 and x_4 are the free variables. So

$$\text{Nul } A = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Subspaces
Summary

- A **subspace** is the same as a span of some number of vectors, but we haven't computed the vectors yet.
- To any matrix is associated two subspaces, the **column space** and the **null space**:

$$\mathrm{Col}\,A = \text{the span of the columns of } A$$
$$\mathrm{Nul}\,A = \text{the solution set of } Ax = 0.$$

How do you check if a subset is a subspace?

- Is it a span? Can it be written as a span?
- Can it be written as the column space of a matrix?
- Can it be written as the null space of a matrix?
- Is it all of $\mathbf{R}^n$ or the zero subspace $\{0\}$?
- Can it be written as a type of subspace that we'll learn about later (eigenspaces, ...)?

If so, then it's automatically a subspace.

If all else fails:

- Can you verify directly that it satisfies the three defining properties?

Linear Dependence and Free Variables

Theorem
Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbf{R}^n$, and consider the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

Then you can delete the columns of A *without* pivots (the columns corresponding to free variables), without changing Span$\{v_1, v_2, \ldots, v_p\}$. The pivot columns are linearly independent, so you can't delete any more columns.

This means that each time you add a pivot column, then the span increases.

$v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5$

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

A has 3 pivot columns: column 1, column 2, & column 5.

Q3:
Example: Which of the vectors in the set {v1,v2,v3,v4,v5} can be removed without changing the span of the vectors?

v1=[1;0;0], v2=[1;1;0], v3=[2;2;0], v4=[1;3;0], v5=[2;2;3]

Q1: describe column space geometrically

Q2: Are the columns of A linearly independent/dependent?

$A_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ Col(A_1) is a line in $\mathbb{R}^3$ (the x_1-axis)

$A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ Col(A_2) is a plane in $\mathbb{R}^3$ (a tilted plane)

$A_3 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ Col(A_3) is a plane in $\mathbb{R}^3$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \right\}$ are lin dep.

$\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ is in Span$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$A_4 = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Col(A_4) is a plane in $\mathbb{R}^3$

$A_5 = A = \begin{bmatrix} 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$ Col(A) is a 3-space in $\mathbb{R}^3$ (it's $\mathbb{R}^3$)

$[A|b] = \left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 2 & b_1 \\ 0 & 1 & 2 & 3 & 2 & b_2 \\ 0 & 0 & 0 & 0 & 3 & b_3 \end{array}\right]$ So Col$(A) = \mathbb{R}^3$

b always in Col A ✓

Sections 2.7 and 2.9

Basis, Dimension, Rank and Basis Theorems

Subspaces
Reminder

Recall: a subspace of $\mathbf{R}^n$ is the same thing as a span, except we haven't computed a spanning set yet.

For example, Col A and Nul A for a matrix A.

There are lots of choices of spanning set for a given subspace.

Are some better than others?

how you choose to represent a subspace as a span of vectors is NEVER unique!

Basis of a Subspace

What is the *smallest number* of vectors that are needed to span a subspace?

Definition
Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in V such that:

1. $V = \text{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

The number of vectors in a basis is the **dimension** of V, and is written dim V.

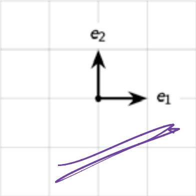
Note the big red border here

Important
A subspace has *many different* bases, but they all have the same number of vectors.

Bases of $\mathbf{R}^2$

Question
What is a basis for $\mathbf{R}^2$?

If ① $\mathbb{R}^2 = \text{Span}\{v_1, v_2\}$
where ② $\{v_1, v_2\}$ are linearly ind.
Then $\{v_1, v_2\}$ are a basis for $\mathbb{R}^2$

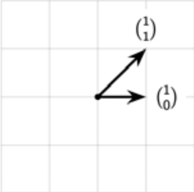


Ex. $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ ← the STANDARD basis for $\mathbb{R}^2$

① $x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$ always consistent, so $\text{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} = \mathbb{R}^2$

② $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ has 2 pivots
so $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ are lin. ind.

FACT: $\mathbb{R}^n$ has dimension equal to n.
So any basis for $\mathbb{R}^n$ has exactly n vectors in it.
e.g. $\left\{\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}, \ldots, \begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}\right\}$ ← e_1, ← e_2, ← e_n



$\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\}$ also a basis for $\mathbb{R}^2$

① $x_1\begin{bmatrix}1\\1\end{bmatrix} + x_2\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$ always consistent
So $\text{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\} = \mathbb{R}^2$

② $\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 1\\0 & -1\end{bmatrix}$ has 2 pivots
so $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\}$ are lin. ind.

Bases of $\mathbf{R}^n$

The unit coordinate vectors

$$e_1 = \begin{pmatrix}1\\0\\\vdots\\0\\0\end{pmatrix}, \quad e_2 = \begin{pmatrix}0\\1\\\vdots\\0\\0\end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix}0\\0\\\vdots\\1\\0\end{pmatrix}, \quad e_n = \begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}$$

are a basis for $\mathbf{R}^n$. The identity matrix has columns $e_1, e_2, \ldots, e_n$.

1. They span: I_n has a pivot in every row.
2. They are linearly independent: I_n has a pivot in every column.

$I_3 = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ The columns of I_3 are the Standard basis for $\mathbb{R}^3$.

$e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

In general: $\{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbf{R}^n$ if and only if the matrix

$$A = \begin{pmatrix}| & | & & |\\v_1 & v_2 & \cdots & v_n\\| & | & & |\end{pmatrix}$$ size $n \times n$

has a pivot in every row and every column.

Basis of a Subspace
Example

Example
Let
$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \mid x + 3y + z = 0 \right\} \qquad \mathcal{B} = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}.$$
Verify that $\mathcal{B}$ is a basis for V. (So dim $V = 2$: it is a *plane*.)

https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=-3,1,0&v2=0,1,-3&range=5&captions=combo

To check that $\mathcal{B} = \left\{\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}\right\}$ is a basis for V

we need to check:

(1) $\text{Span}\left\{\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}\right\} = V$

(2) $\left\{\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}\right\}$ is linearly ind

Check (2)

$$\begin{bmatrix} -3 & 0 \\ 1 & 1 \\ 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} (1) & 0 \\ 0 & (1) \\ 0 & 0 \end{bmatrix}$$ 2 pivots

So $\left\{\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}\right\}$ are lin ind. ✓

(1) try ad hoc method

$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is in V if $x + 3y + z = 0 \Rightarrow z = -x - 3y$?? ✓

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ -x-3y \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ -x \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ -3y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$
$$= x\left(-\tfrac{1}{3}\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + \tfrac{1}{3}\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}\right) + y\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$
$$= -\tfrac{1}{3}x\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + \left(\tfrac{1}{3}x + y\right)\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

Note $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$

So $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is in $\text{span}\left\{\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}\right\}$

is in $\text{Span}\left\{\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}\right\}$

Q: What is another basis for $V = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + 3y + z = 0\right\}$

The set V is just all solutions to the system $\{x + 3y + z = 0$.

y z free

$V = \text{Nul}(A)$ $A = \begin{bmatrix} 1 & 3 & 1 \end{bmatrix}$

$$\begin{aligned} x + 3y + z &= 0 \\ y &= y \text{ (Free)} \\ z &= z \text{ (Free)} \end{aligned} \Rightarrow \begin{aligned} x &= -3y - z \\ y &= y \\ z &= z \end{aligned} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So. $V = \text{span}\left\{\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$

and $\left\{\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ are lin ind

So they form a BASIS for V.

Example: Find a basis for Nul(A) with A given below (the RREF of A is also given)

Basis for Nul A

Example
$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Fact
The vectors in the parametric vector form of the general solution to $Ax = 0$ always form a basis for Nul A.

Example: Find a basis for Nul(A) with A given below
(the RREF of A is also given)

Nul A = {x | Ax = 0} all solutions to homogeneous Ax = 0.

(those x's → Ax = 0)

Basis for Nul A

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(x_3, x_4 free)

Fact

The vectors in the parametric vector form of the general solution to $Ax = 0$ always form a basis for Nul A.

$$\begin{cases} x_1 = 8x_3 + 7x_4 \\ x_2 = -4x_3 - 3x_4 \\ x_3 = x_3 \text{ (free)} \\ x_4 = x_4 \text{ (free)} \end{cases} \Rightarrow X = x_3 \begin{bmatrix} 8 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

So solutions to $Ax = 0$ are in

$$\text{Span}\left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{bmatrix} 7 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

& $\left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{bmatrix} 7 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ are lin ind.

So they are a basis for Nul(A)

Summary so far.

A basis for subspace V is a set of vectors $\{v_1, v_2, v_3\}$ s.t.

① $\{v_1, v_2, v_3\}$ span the subspace V ← find them for Nul A: get parametric vector form

② $\{v_1, v_2, v_3\}$ are lin ind. ← row reduce & count pivots

OR for two vectors just check if they are scalar multiples of each other.

Example: find a basis for Col(A) for the matrix A given below (the RREF of A is also given)

Basis for Col A

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns = basis ⟷ pivot columns in rref

Fact
The pivot columns of A always form a basis for Col A.

Warning: I mean the pivot columns of the *original* matrix A, not the row-reduced form. (Row reduction changes the column space.)

We are looking for some set of vectors $\{v_1, v_2\}$ (why 2?)

s.t.

① $\operatorname{Col} A = \operatorname{Span}\{v_1, v_2\}$

② $\{v_1, v_2\}$ should be linearly ind.

NOTE

$$\operatorname{Col} A = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ -2 \end{pmatrix} \right\}$$

↑ in general $A = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$ (size 3x4)

Then Col A could be (a priori)

(0) only $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

(1) a line in $\mathbb{R}^3$

(2) a plane in $\mathbb{R}^3$

(3) all of $\mathbb{R}^3$

IDEA: just extract the pivot cols of the ORIGINAL MATRIX & these will be a basis for Col A.

So $\left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \right\}$ are a basis for Col A

Check: ② ✓ linearly ind $\begin{bmatrix} 1 & 2 \\ -2 & -3 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ 2 pivots.

① (i) $\begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$ is already in $\operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \right\}$

(ii) $\begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}$ is already in $\operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \right\}$

didn't need a pivot for col 3 or col 4.

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns = basis ⟷ pivot columns in rref

If (i) & (ii) both true then

$$\operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \right\} = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ -2 \end{pmatrix} \right\} = \operatorname{Col} A$$

The Basis Theorem

Basis Theorem
Let V be a subspace of dimension m. Then:

- Any m linearly independent vectors in V form a basis for V.
- Any m vectors that span V form a basis for V.

Example: any three linearly independent vectors form a basis for $\mathbb{R}^3$.

Upshot
If you *already* know that $\dim V = m$, and you have m vectors $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ in V, then you only have to check *one of*

1. $\mathcal{B}$ is linearly independent, *or*
2. $\mathcal{B}$ spans V

in order for $\mathcal{B}$ to be a basis.

The Rank Theorem

Recall:

- The **dimension** of a subspace V is the number of vectors in a basis for V.
- A basis for the column space of a matrix A is given by the pivot columns. ← NOT the columns of the REF/RREF (you'll have the right number of vectors but the WRONG SPAN!)
- A basis for the null space of A is given by the vectors attached to the free variables in the parametric vector form.

Definition

The **rank** of a matrix A, written rank A, is the dimension of the column space Col A. The **nullity** of A, written nullity A, is the dimension of the solution set of $Ax = 0$.

$\text{rank}(A) = \#\text{ pivots} = \dim \text{Col } A.$

$\text{nullity}(A) = \#\text{ free vars} = \dim \text{Nul } A.$

Rank Theorem

If A is an $m \times n$ matrix, then

$$\text{rank } A + \text{nullity } A = n = \text{the number of columns of } A.$$

so that is:

$\#\text{ pivots of } A + \#\text{ free vars of } A = \#\text{ cols of } A.$

(dimension of column space) + (dimension of solution set to Ax=0) = (number of variables)

Proof: Every column is either pivot or free (& not both).

The Rank Theorem

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

basis of Col A — free variables

$\text{rank } A = 2 = \dim \text{Col } A$

$\text{nullity } A = 2 = \dim \text{Nul } A$

$2 + 2 = 4 = \#\text{ cols}$

You Try It!

Poll

Poll

True or False: If A is a 10×15 matrix and there is a basis of Col A consisting of 4 vectors, then there is a basis of Nul A consisting of 6 vectors.

False

A has 10 rows, 15 cols.

So if $\dim \text{Col } A = \text{rank } A = 4$

then A has 4 pivot cols

so A has 11 free cols

i.e. $\dim \text{Nul } A = 11$ (not 6).

Summary

- A **basis** of a subspace is a minimal set of spanning vectors.
- There are recipes for computing a basis for the column space and null space of a matrix. ≠ NOT THE BASIS.
- The **dimension** of a subspace is the number of vectors in any basis.
- The **basis theorem** says that if you already know that $\dim V = m$, and you have m vectors in V, then you only have to check if they span *or* they're linearly independent to know they're a basis.
- The **rank theorem** says the dimension of the column space of a matrix, plus the dimension of the null space, is the number of columns of the matrix.

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns = basis ⟵ pivot columns in rref

extract THESE columns for a basis for Col A

NOT these columns!

Ex. Find a basis for $\text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \right\}$

META: ① write $A = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

② row reduce to REF (don't need RREF) to find pivots

③ extract pivot cols of ORIGINAL MATRIX. (not the REF/RREF!)

extract this one.

don't extract this column

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So $\text{Col}\, A = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \right\} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

$\neq \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$