Chapter 3

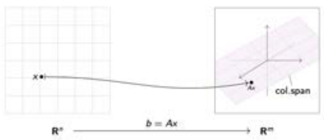
Linear Transformations and Matrix Algebra

Section 3.1

Matrix Transformations

Motivation

Let A be a matrix, and consider the matrix equation $b = Ax$. If we vary x, we can think of this as a *function* of x.



Good news & bad news:

In this class there are (at least) TWO different reasons for a topic/concept to be hard:

① computations / technical difficulty

② connections between ideas/ theoretical diff'ty

Example: A=[1 0 ; 0 1 ; 0 0], plot some inputs x and outputs Ax

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

① For some x's in $\mathbb{R}^2$, compute Ax.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -3 \\ 4 \end{bmatrix} = -3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}$$

So,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$$

Example: with A as below, plot some inputs and outputs

Matrices as Functions

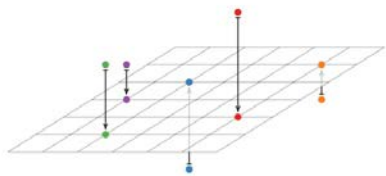
Projection

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?mat=1,0,0:0,1,0:0,0,0&range2=5&closed=true

Now find some "random" inputs x in $\mathbb{R}^3$ & compute the output Ax.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2+0+0 \\ 0+3+0 \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$



[interactive]

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

So,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

Example: for the matrix A given below, plot some input and output vectors for T(x)=Ax, and give the function T an appropriate NAME.

https://textbooks.math.gatech.edu/ila/demos/twobytwo.html?mat=-1,0,0,1&closed=true

Matrices as Functions
Reflection

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

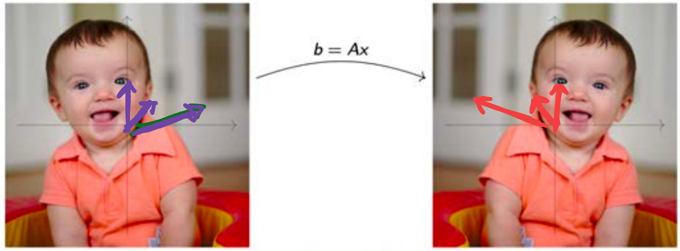
Input Output

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+0 \\ 0+1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ b \end{bmatrix}$$ Name?



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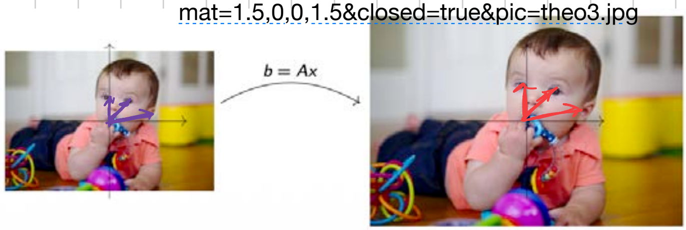
name: horizontal reflection, y-axis reflection, transformation, x-value negativizer, sideways flip!

Example: plot some input and output vectors for T(x)=Ax and give the function T a NAME.

https://textbooks.math.gatech.edu/ila/demos/twobytwo.html?mat=1.5,0,0,1.5&closed=true&pic=theo3.jpg

Matrices as Functions
Dilation

$$A = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}$$



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$$\begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}$$

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} (3)(1.5)+0 \\ 0+(1)(1.5) \end{bmatrix} = \begin{bmatrix} 4.5 \\ 1.5 \end{bmatrix}$$

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2(1.5) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1.5a \\ 1.5b \end{bmatrix}$$

Names?

Tim (for T), The One-and-a-half-er

xy-stretcher

watch bubbles at 1.5x speed.

Example: ditto

Matrices as Functions
Identity

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

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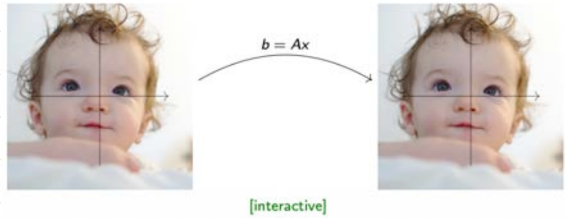


[interactive]

Name

Samenator, the do-nothing

The twin transformation.

Matrices as Functions
Rotation

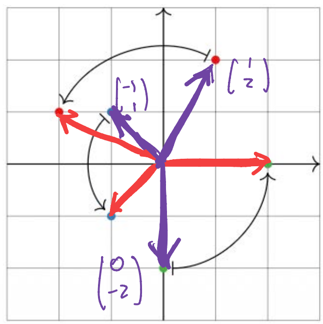
$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix}$$



Names: add 90°, rotate 90°,

Spin 90°

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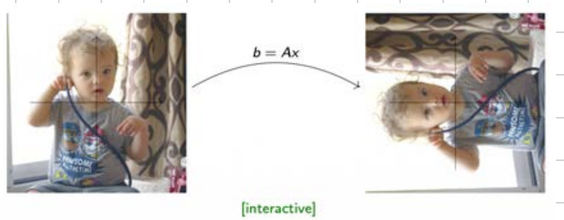

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Other Geometric Transformations

In §3.1 there are other examples of geometric transformations of $\mathbf{R}^2$ given by matrices. Please look them over.

What just happened?

I want to convince you that if A is a matrix ($m \times n$)

m rows, n columns.

and x is some input vector x in $\mathbb{R}^n$

then Ax is some new vector b in $\mathbb{R}^m$

and moving x to other vectors in $\mathbb{R}^n$ gives you new output vectors b in $\mathbb{R}^m$

The matrix A is doing something to input vectors to get output vectors

Transformations

Motivation

We have been drawing pictures of what it looks like to multiply a matrix by a vector, as a function of the vector.

Now let's go the other direction. Suppose we have a function, and we want to know, does it come from a matrix?

Example

For a vector x in $\mathbf{R}^2$, let $T(x)$ be the counterclockwise rotation of x by an angle θ. Is $T(x) = Ax$ for some matrix A?

If $\theta = 90°$, then we know $T(x) = Ax$, where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

But for general θ, it's not clear.

Our next goal is to answer this kind of question.

Transformations

Vocabulary

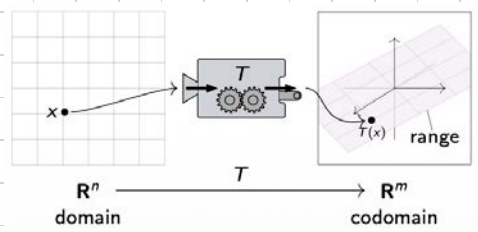
Definition

A **transformation** (or **function** or **map**) from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule T that assigns to each vector x in $\mathbf{R}^n$ a vector $T(x)$ in $\mathbf{R}^m$.

- $\mathbf{R}^n$ is called the **domain** of T (the inputs).
- $\mathbf{R}^m$ is called the **codomain** of T (where the outputs live).
- For x in $\mathbf{R}^n$, the vector $T(x)$ in $\mathbf{R}^m$ is the **image** of x under T. Notation: $x \mapsto T(x)$.
- The set of all images $\{T(x) \mid x \text{ in } \mathbf{R}^n\}$ is the **range** of T.

Notation:

$T: \mathbf{R}^n \longrightarrow \mathbf{R}^m$ means T is a transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$.



Examples from calculus: f(x)=sin(x) and g(x)=x^2

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = \sin x$

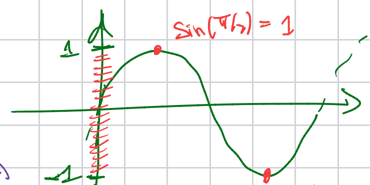
domain is $\mathbb{R}$ (where inputs live)

codomain is $\mathbb{R}$ (where outputs are)

range is $[-1, 1]$

1 is an output

$\sin(\pi/2) = 1$



$\sin\left[\frac{3\pi}{2}\right] = -1$

-1 is an output

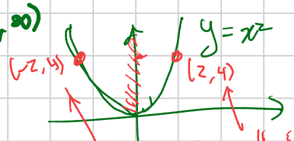
$g: \mathbb{R} \to \mathbb{R}$

$g(x) = x^2$

domain is $\mathbb{R}$ (where inputs live)

codomain is $\mathbb{R}$ (where outputs are)

range is $[0, \infty)$

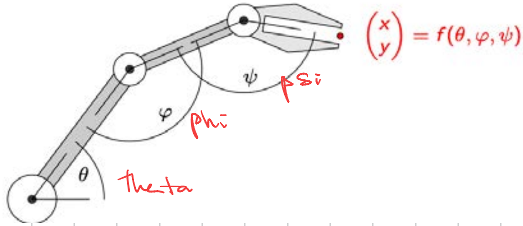


4 is an output

4 is also an output for -2.

Functions from Engineering

Suppose you are building a robot arm with three joints that can move its hand around a plane, as in the following picture.



Matrix Transformations

Definition
Let A be an $m \times n$ matrix. The **matrix transformation** associated to A is the transformation

$$T : \mathbf{R}^n \longrightarrow \mathbf{R}^m \quad \text{defined by} \quad T(x) = Ax.$$

Example: find the domain, codomain, and range of the matrix transformation T(x) below

$$T\begin{pmatrix}-1\\-2\\-3\end{pmatrix} = \begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}\begin{pmatrix}-1\\-2\\-3\end{pmatrix} = \begin{pmatrix}-14\\-32\end{pmatrix}.$$

where inputs live

- The *domain* of T is $\mathbf{R}^n$, which is the number of *columns* of A.
- The *codomain* of T is $\mathbf{R}^m$, which is the number of *rows* of A.
- The *range* of T is the set of all images of T:

$$T(x) = Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}\begin{pmatrix}x_1\\x_2\\ \vdots \\ x_n\end{pmatrix} = x_1v_1 + x_2v_2 + \cdots + x_nv_n.$$

where outputs live.

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$$

① domain is $\mathbb{R}^3$
② codomain is $\mathbb{R}^2$
③ range is ???

$$= x\begin{bmatrix}1\\4\end{bmatrix} + y\begin{bmatrix}2\\5\end{bmatrix} + z\begin{bmatrix}3\\6\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$$

$\begin{bmatrix}b_1\\b_2\end{bmatrix}$ is in the range exactly when it is a linear combination of the columns of A.

range of T is Span$\left\{\begin{bmatrix}1\\4\end{bmatrix}, \begin{bmatrix}2\\5\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix}\right\}$.

Example: find the domain, codomain, and range of the matrix transformation T(x)=Ax with the matrix A given below

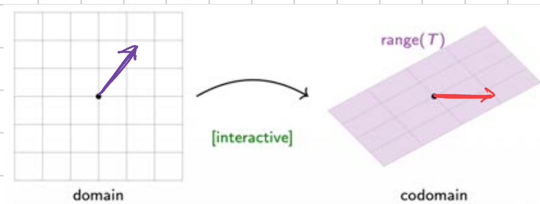
$$A = \begin{pmatrix}-1 & 0\\ 2 & 1\\ 1 & -1\end{pmatrix}$$

2 is # of cols

3 is # of rows

① domain is $\mathbb{R}^2$
② codomain is $\mathbb{R}^3$
③ range is span$\left\{\begin{bmatrix}-1\\2\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$

https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?mat=-1,0:2,1:1,-1&closed=true&show=false



You Try It!

Poll

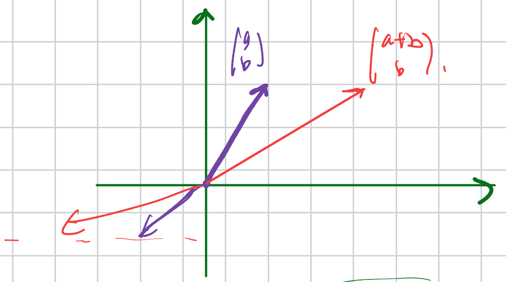
Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T : \mathbf{R}^2 \to \mathbf{R}^2$. ($T$ is called a **shear**.)

Find: the domain, the codomain, and the range of T.

Also plot a few input and output vectors, and give the transformation a NAME

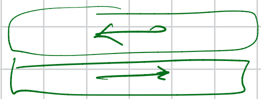
domain is $\mathbb{R}^2$
codomain is $\mathbb{R}^2$
range is span$\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ b \end{bmatrix}$$



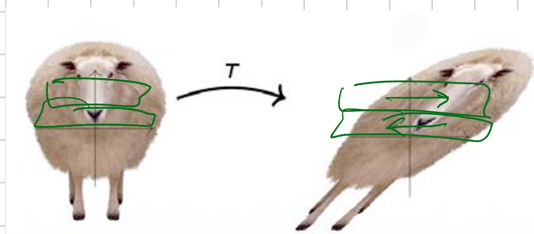
Name: x-~~increaser~~ changer

Shear.



Summary

- We can think of $b = Ax$ as a **transformation** with input x and output b.
- There are vocabulary words associated to transformations: **domain**, **codomain**, **range**.
- A transformation that comes from a matrix is called a **matrix transformation**.
- In this case, the vocabulary words mean something concrete in terms of matrices.
- We like transformations that come from matrices, because questions about those transformations turn into questions about matrices.



Section 3.2

One-to-one and Onto Transformations

Matrix Transformations
Reminder

Recall: Let A be an $m \times n$ matrix. The **matrix transformation** associated to A is the transformation

$$T : \mathbf{R}^n \longrightarrow \mathbf{R}^m \quad \text{defined by} \quad T(x) = Ax.$$

- The *domain* of T is $\mathbf{R}^n$, which is the number of *columns* of A.
- The *codomain* of T is $\mathbf{R}^m$, which is the number of *rows* of A.
- The *range* of T is the set of all images of T:

$$T(x) = Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1v_1 + x_2v_2 + \cdots + x_nv_n.$$

This is the *column space* of A. It is a span of vectors in the codomain.

Matrix Transformations
Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T : \mathbf{R}^2 \to \mathbf{R}^3$.

(1) find T(u) where u=[3;4]

(2) let b=[7;5;7], find v in R^2 such that T(v)=b.

(3) is there any other vector w (other than v) that also gets sent to b?

① $T\left(\begin{bmatrix}3\\4\end{bmatrix}\right) = \begin{bmatrix}1&1\\0&1\\1&1\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}7\\4\\7\end{bmatrix}$

$\begin{bmatrix}7\\4\\7\end{bmatrix}$ is the image of $\begin{bmatrix}3\\4\end{bmatrix}$

or $\begin{bmatrix}3\\4\end{bmatrix}$ maps to $\begin{bmatrix}7\\4\\7\end{bmatrix}$.

② Find $v = \begin{bmatrix}x\\y\end{bmatrix}$ such that

$Av = \begin{bmatrix}7\\5\\7\end{bmatrix}$?

aka $\begin{bmatrix}1&1\\0&1\\1&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}7\\5\\7\end{bmatrix}$?

$\iff x\begin{bmatrix}1\\0\\1\end{bmatrix} + y\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}7\\5\\7\end{bmatrix}$

$\iff \begin{bmatrix}x+y\\y\\x+y\end{bmatrix} = \begin{bmatrix}7\\5\\7\end{bmatrix}$

$\iff \begin{cases} x+y=7 \\ y=5 \\ x+y=7 \end{cases}$

$\iff \left[\begin{array}{cc|c}1&1&7\\0&1&5\\1&1&7\end{array}\right]$ row reduce THIS!

② cont'.

row reduce $[A|b]$.

$[A|b] = \left[\begin{array}{cc|c}1&1&7\\0&1&5\\1&1&7\end{array}\right]$

$\sim \left[\begin{array}{cc|c}1&1&7\\0&1&5\\0&0&0\end{array}\right]$

$\sim \left[\begin{array}{cc|c}1&0&2\\0&1&5\\0&0&0\end{array}\right]$ $\quad x=2$, $y=5$

$v = \begin{bmatrix}2\\5\end{bmatrix}$ i.e. $T\left(\begin{bmatrix}2\\5\end{bmatrix}\right) = \begin{bmatrix}7\\5\\7\end{bmatrix}$

The image of $\begin{bmatrix}2\\5\end{bmatrix}$ is $\begin{bmatrix}7\\5\\7\end{bmatrix}$.

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T : \mathbf{R}^2 \to \mathbf{R}^3$.

Translation: Find c such that $Ax = c$ is inconsistent.

Same A as previous example.

(4) is there any vector c in R^3 which is not in the range of T? In other words, can you find a vector c for which NO vector v in R^2 maps to c?

Is it possible that $Ax = c$ is inconsistent for some choice of c?

possible for pivot to be in augmented column?

$$[A \mid c] = \left[\begin{array}{cc|c} 1 & 1 & c_1 \\ 0 & 1 & c_2 \\ 1 & 1 & c_3 \end{array}\right]$$

(note: if $c_1 \neq c_3$ we have a problem)

Try $c = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

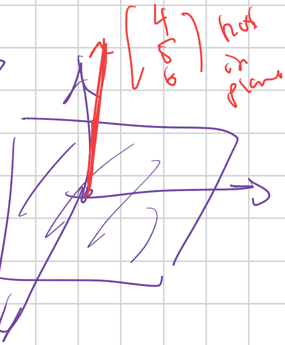
$$[A \mid c] = \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 1 & 5 \\ 1 & 1 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{array}\right]$$

pivot in aug. col!

So $Ax = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ is inconsistent.

No x maps to $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$.

IDEA. range of T is $\text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$



Matrix Transformations
Non-Example

Note: All of these questions are questions about *the transformation* T; it still makes sense to ask them in the absence of the matrix A.

The fact that T comes from a matrix means that these questions translate into questions about a matrix, which we know how to do.

Non-example: $T: \mathbf{R}^2 \to \mathbf{R}^3$ $\quad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin x \\ xy \\ \cos y \end{pmatrix}$

We will restrict our focus to matrix transformations in this class, but it is good to know that you can still ask similar questions about arbitrary transformations from R^n to R^m.

Questions About Transformations

Today we will focus on two important questions one can ask about a transformation $T: \mathbf{R}^n \to \mathbf{R}^m$:

- Do there exist distinct vectors x, y in $\mathbf{R}^n$ such that $T(x) = T(y)$?
- For every vector v in $\mathbf{R}^m$, does there exist a vector x in $\mathbf{R}^n$ such that $T(x) = v$?

These are subtle because of the multiple *quantifiers* involved ("for every", "there exists").

One-to-one Transformations

Definition

A transformation $T: \mathbf{R}^n \to \mathbf{R}^m$ is **one-to-one** (or **into**, or **injective**) if different vectors in $\mathbf{R}^n$ map to different vectors in $\mathbf{R}^m$. In other words, for every b in $\mathbf{R}^m$, the equation $T(x) = b$ has *at most one* solution x. Or, different inputs have different outputs. Note that *not* one-to-one means at least two different vectors in $\mathbf{R}^n$ have the same image.

Exam 2 in about a week on Friday June 26

Matrix Transformations
Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T: \mathbf{R}^2 \to \mathbf{R}^3$.

Same example as before: Is the transformation T(x)=Ax one-to-one?

If $Ax=b$ for some b then x is uniquely determined (b/c no free var)

Characterization of One-to-One Matrix Transformations

Theorem

Let $T: \mathbf{R}^n \to \mathbf{R}^m$ be a matrix transformation with matrix A. Then the following are equivalent:

- T is one-to-one.
- For each b in $\mathbf{R}^m$, the equation $T(x) = b$ has at most one solution.
- For each b in $\mathbf{R}^m$, the equation $Ax = b$ has a unique solution or is inconsistent.
- $Ax = 0$ has a unique solution. ($x=0$ trivial soln)
- The columns of A are linearly independent.
- A has a pivot in every column.

Question

If $T: \mathbf{R}^n \to \mathbf{R}^m$ is one-to-one, what can we say about the relative sizes of n and m?

What can you say about A? $n \leq m$ (# cols $\leq$ # rows)

What sizes can A be? can be TALL

What sizes can A NOT be?, can NOT be WIDE!

New example:

One-to-One Transformations
Example

Define

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad T(x) = Ax,$$

so $T : \mathbb{R}^2 \to \mathbb{R}^3$. Is T one-to-one?

(1) domain of T is $\mathbb{R}^2$

(2) codomain of T is $\mathbb{R}^3$

(3) range of T is a plane in $\mathbb{R}^3$, (range is Span$\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$)

(4) is T one-to-one? yes why? b/c A has a pivot in every column.

One-to-one Transformations

Definition

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **one-to-one** (or **into**, or **injective**) if different vectors in $\mathbb{R}^n$ map to different vectors in $\mathbb{R}^m$. In other words, for every b in $\mathbb{R}^m$, the equation $T(x) = b$ has *at most one* solution x. Or, different inputs have different outputs. Note that *not* one-to-one means at least two different vectors in $\mathbb{R}^n$ have the same image.

Non-example:

One-to-One Transformations
Non-Example

Define

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad T(x) = Ax,$$

(1) Is the transformation T(x)=Ax one-to-one?

(1) $\boxed{\text{No}}$ why? not a pivot in 3rd column of A.

(2) If not, find two **different** vectors v and w that map to the **same** output b

(3) Also, is **every** vector b in R^2 the output for some input vector v?

(2) Soln. Solve $Ax=0$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

x_3

$$x = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Set $x_3 = 1$ get $x = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Easy way Find two vectors that map to $\begin{bmatrix}0\\0\end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $\begin{bmatrix}0\\0\\0\end{bmatrix}$ maps to $\begin{bmatrix}0\\0\end{bmatrix}$

Check

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So $T\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\-1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}$ but inputs $\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}$ are different

So $\boxed{T \text{ is not one-to-one.}}$ (2)

(3) If $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ arbitrary in $\mathbb{R}^2$ then is $T(x) = Ax = b$ consistent? (always an x soln?)

row reduce $\left(\begin{array}{ccc|c} 1 & 1 & 0 & b_1 \\ 0 & 1 & 1 & b_2 \end{array}\right)$ no pivot in aug col.

So $Ax=b$ always consistent $\boxed{(3)\text{ yes}}$

Onto Transformations

Definition

A transformation $T: \mathbf{R}^n \to \mathbf{R}^m$ is **onto** (or **surjective**) if the range of T is equal to $\mathbf{R}^m$ (its codomain). In other words, for every b in $\mathbf{R}^m$, the equation $T(x) = b$ has at *least one solution*. Or, every possible output has an input. Note that *not* onto means there is some b in $\mathbf{R}^m$ which is not the image of any x in $\mathbf{R}^n$.

Characterization of Onto Matrix Transformations

Theorem

Let $T: \mathbf{R}^n \to \mathbf{R}^m$ be a matrix transformation with matrix A. Then the following are equivalent:

- T is onto
- $T(x) = b$ has a solution for every b in $\mathbf{R}^m$
- $Ax = b$ is consistent for every b in $\mathbf{R}^m$
- The columns of A span $\mathbf{R}^m$
- A has a pivot in every row

Same example as before:

One-to-One Transformations

Non-Example

Define

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad T(x) = Ax,$$

(1) is T(x)=Ax onto? is every b in $\mathbb{R}^2$ in the range of T?

yes. why $Ax=b$ can't be inconsistent?

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & b_1 \\ 0 & 1 & 1 & b_2 \end{array}\right]$$

Could have failed if

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & * \\ 0 & 0 & 0 & * \end{array}\right]$$

non-zero

Same example as before:

Matrix Transformations

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T: \mathbf{R}^2 \to \mathbf{R}^3$.

(1) is T(x)=Ax onto?

range of T is span$\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ is a plane in $\mathbb{R}^3$

So lets find the pivots.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

T is NOT ONTO b/c A doesn't have a pivot in EVERY row

(A has a pivot in row 1 & row 2 only)

(2) Find b not in range of $T(x)=Ax$.

try $b = \begin{bmatrix}1\\1\\0\end{bmatrix}$?

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{array}\right]$$

So $\begin{bmatrix}1\\1\\0\end{bmatrix}$ is NOT in span of cols of A so not in range of T.

Question

If $T: \mathbf{R}^n \to \mathbf{R}^m$ is onto, what can we say about the relative sizes of n and m?

* If A is $m \times n$ then $m \le n$ (# rows ≤ # cols)

* A can be WIDE for T to be onto

* A can NOT be tall for T to be onto.

Last example:

One-to-One and Onto Transformations

Non-Example

Define

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \qquad T(x) = Ax,$$

so $T: \mathbf{R}^3 \to \mathbf{R}^2$. Is T one-to-one? Is it onto?

(1) find the domain and codomain of T

domain is $\mathbb{R}^3$, codomain is $\mathbb{R}^2$

range of T is span $\left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix} \right\}$

$$\begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

$\begin{bmatrix} 2 \\ -4 \end{bmatrix}$ is in the range of T.

(or about $\begin{bmatrix} \vdots \end{bmatrix}$)

(2) is T one-to-one?

(3) is T onto?

count pivots in A.

one pivot in col 1 & row 1,

two free columns

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

2nd row has no pivot

(2) is T one-to-one? NO. there are free cols.

(3) is T onto? No, not every row has a pivot.

The range of T is a line in $\mathbb{R}^2$

(not all of $\mathbb{R}^2$)

Summary

- A transformation T is **one-to-one** if $T(x) = b$ has *at most one* solution, for every b in $\mathbf{R}^m$.
- A transformation T is **onto** if $T(x) = b$ has *at least one* solution, for every b in $\mathbf{R}^m$.
- A matrix transformation with matrix A is one-to-one if and only if the columns of A are linearly independent, if and only if A has a pivot in every column.
- A matrix transformation with matrix A is onto if and only if the columns of A span $\mathbf{R}^m$, if and only if A has a pivot in every row.
- Two of the most basic questions one can ask about a transformation is whether it is one-to-one or onto.

Section 3.3

Linear Transformations

Linear Transformations
Motivation

In the last two lectures we have been asking questions about transformations, and answering them in the case of matrix transformations.

However, sometimes it is not clear if a transformation is a matrix transformation or not.

Example

For a vector x in $\mathbf{R}^2$, let $T(x)$ be the counterclockwise rotation of x by an angle θ. Is $T(x) = Ax$ for some matrix A?

Key observation:

Recall: If A is a matrix, u, v are vectors, and c is a scalar, then

$$A(u+v) = Au + Av \qquad A(cv) = cAv.$$

So if $T(x) = Ax$ is a matrix transformation then,

$$T(u+v) = T(u) + T(v) \quad \text{and} \quad T(cv) = cT(v).$$

Linear Transformations

Definition

A transformation $T: \mathbf{R}^n \to \mathbf{R}^m$ is **linear** if it satisfies the equations for all vectors u, v in $\mathbf{R}^n$ and all scalars c.

(1) $T(u+v) = T(u) + T(v)$ "T respects addition"

(2) $T(cv) = c\,T(v)$ "T respects scalar multiplication"

Some immediate consequences:

Check: if T is linear, then

$$T(0) = 0 \qquad T(cu + dv) = cT(u) + dT(v)$$

for all vectors u, v and scalars c, d. More generally,

$$T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n).$$

In engineering this is called **superposition**.

Linear Transformations
Dilation

Define $T: \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = 1.5x$. Is T linear? Check:

Note: T is a matrix transformation!

$$T(x) = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix} x,$$

as we checked before.

Example $T\left(\begin{pmatrix}1\\2\end{pmatrix}\right) = 1.5\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1.5\\3\end{pmatrix}$

$$\boxed{T\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = 1.5\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}1.5a\\1.5b\end{bmatrix}}$$

(1) $T\left(\begin{bmatrix}a\\b\end{bmatrix} + \begin{bmatrix}c\\d\end{bmatrix}\right) = T\left(\begin{bmatrix}a+c\\b+d\end{bmatrix}\right) = \begin{pmatrix}1.5(a+c)\\1.5(b+d)\end{pmatrix} = \begin{pmatrix}1.5a\\1.5b\end{pmatrix} + \begin{pmatrix}1.5c\\1.5d\end{pmatrix}$ ✓

$= T\left(\begin{bmatrix}a\\b\end{bmatrix}\right) + T\left(\begin{bmatrix}c\\d\end{bmatrix}\right)$

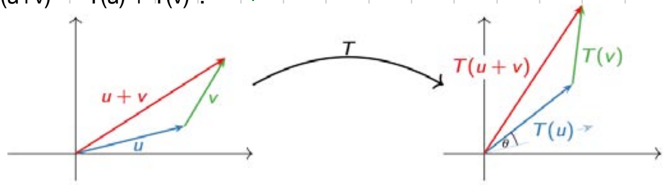
(2) $T\left(k\begin{pmatrix}a\\b\end{pmatrix}\right) = T\left(\begin{bmatrix}ka\\kb\end{bmatrix}\right) = 1.5\begin{pmatrix}ka\\kb\end{pmatrix} = k\begin{pmatrix}1.5a\\1.5b\end{pmatrix} = kT\left(\begin{bmatrix}a\\b\end{bmatrix}\right)$ ✓

Linear Transformations
Rotation

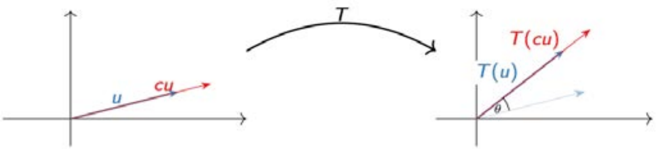
Define $T: \mathbf{R}^2 \to \mathbf{R}^2$ by
$T(x) =$ the vector x rotated counterclockwise by an angle of θ.
Is T linear? Check:

Proof by picture:

(1) $T(u+v) \stackrel{?}{=} T(u) + T(v)$? ✓



(2) $T(cv) \stackrel{?}{=} c\,T(v)$? ✓



Linear Transformations
Non-example

Is every transformation a linear transformation?

No! For instance, $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin x \\ xy \\ \cos y \end{pmatrix}$ is not linear.

Hint: check what happens to T(v) if v=0 the zero vector.

$$T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \sin 0 \\ 0 \cdot 0 \\ \cos 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So T is <u>not</u> linear since for any <u>LINEAR</u> transformation (from $\mathbb{R}^2$ to $\mathbb{R}^3$) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ would need to map to $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

You Try It!

Poll

Which of the following transformations are linear?

A. $T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}|x_1|\\x_2\end{pmatrix}$ $\boxed{\textbf{B. } T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}2x_1+x_2\\x_1-2x_2\end{pmatrix}}$?? ← 1-to-1 onto?

C. $T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1x_2\\x_2\end{pmatrix}$ **D.** $T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}2x_1+1\\x_1-2x_2\end{pmatrix}$

(1) $T(u+v) \stackrel{?}{=} T(u)+T(v)$?

(2) $T(cv) \stackrel{?}{=} cT(v)$?

D No $T\left(\begin{pmatrix}0\\0\end{pmatrix}\right) = \begin{pmatrix}1\\0\end{pmatrix}$

B yes (we will see why in a min)

A & C both no.

(A) $u = \begin{bmatrix}1\\1\end{bmatrix}$, $v = \begin{pmatrix}-1\\1\end{pmatrix}$

$u+v = \begin{pmatrix}0\\2\end{pmatrix}$ $T(u+v) = \begin{bmatrix}0\\2\end{bmatrix}$ different

$T(u)+T(v) = \begin{pmatrix}1\\1\end{pmatrix} + \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}$

so T doesn't respect addition.

(C) try $u = \begin{pmatrix}2\\2\end{pmatrix}$ $v = \begin{pmatrix}3\\3\end{pmatrix}$

$T(u+v) = T\left(\begin{bmatrix}5\\5\end{bmatrix}\right) = \begin{pmatrix}25\\5\end{pmatrix}$ different

$T(u)+T(v) = \begin{pmatrix}4\\2\end{pmatrix} + \begin{bmatrix}9\\3\end{bmatrix} = \begin{pmatrix}13\\5\end{pmatrix}$

so T doesn't respect addition.

The Matrix of a Linear Transformation

We will see that a *linear* transformation T is a matrix transformation: $T(x) = Ax$.

But what matrix does T come from? What is A?

Here's how to compute it.

The Matrix of a Linear Transformation

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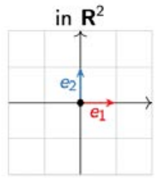
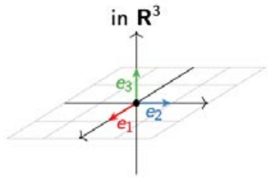
But what matrix does T come from? What is A?

Here's how to compute it.





First: recall the definitions of e1,e2,...,en

Unit Coordinate Vectors

Definition

The **unit coordinate vectors** in $\mathbf{R}^n$ are

This is what $e_1, e_2, \ldots$ mean, for the rest of the class.

$$\boxed{e_1 = \begin{pmatrix}1\\0\\\vdots\\0\\0\end{pmatrix},\quad e_2 = \begin{pmatrix}0\\1\\\vdots\\0\\0\end{pmatrix},\quad \ldots,\quad e_{n-1} = \begin{pmatrix}0\\0\\\vdots\\1\\0\end{pmatrix},\quad e_n = \begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}.}$$

aka Standard basis vectors in $\mathbb{R}^n$.

For the matrix A=[1 2 3 ; 4 5 6 ; 7 8 9] compute Ae1, Ae2, and Ae3.

$$Ae_1 = \begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{bmatrix}1\\4\\7\end{bmatrix}$$

First col of A

Second col of A

$$Ae_2 = \begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{bmatrix}2\\5\\8\end{bmatrix}$$

Third col of A.

$$Ae_3 = \begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{bmatrix}3\\6\\9\end{bmatrix}$$

Note: if A is an $m \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, then $Ae_i = v_i$ for $i = 1, 2, \ldots, n$: multiplying a matrix by e_i gives you the ith column.

Linear Transformations are Matrix Transformations

Theorem

Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Let

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}.$$

This is an $m \times n$ matrix, and T is the matrix transformation for A: $T(x) = Ax$.

The matrix A is called the **standard matrix** for T.

Take-Away

Linear transformations are the same as matrix transformations.

$T(e_1) = Ae_1 =$ First col of A

$T(e_2) = Ae_2 =$ Second col of A

$T(e_3) = Ae_3 =$ Third col of A.

Linear Transformations are Matrix Transformations
Example

Before, we defined a **dilation** transformation $T: \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = 1.5x$.
What is its standard matrix?

How to get A? Apply T to e_1 & e_2

$T(e_1) = T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1.5\\0\end{bmatrix}$ first col of A

$T(e_2) = T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1.5\end{bmatrix}$ second col of A

$$A = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}$$

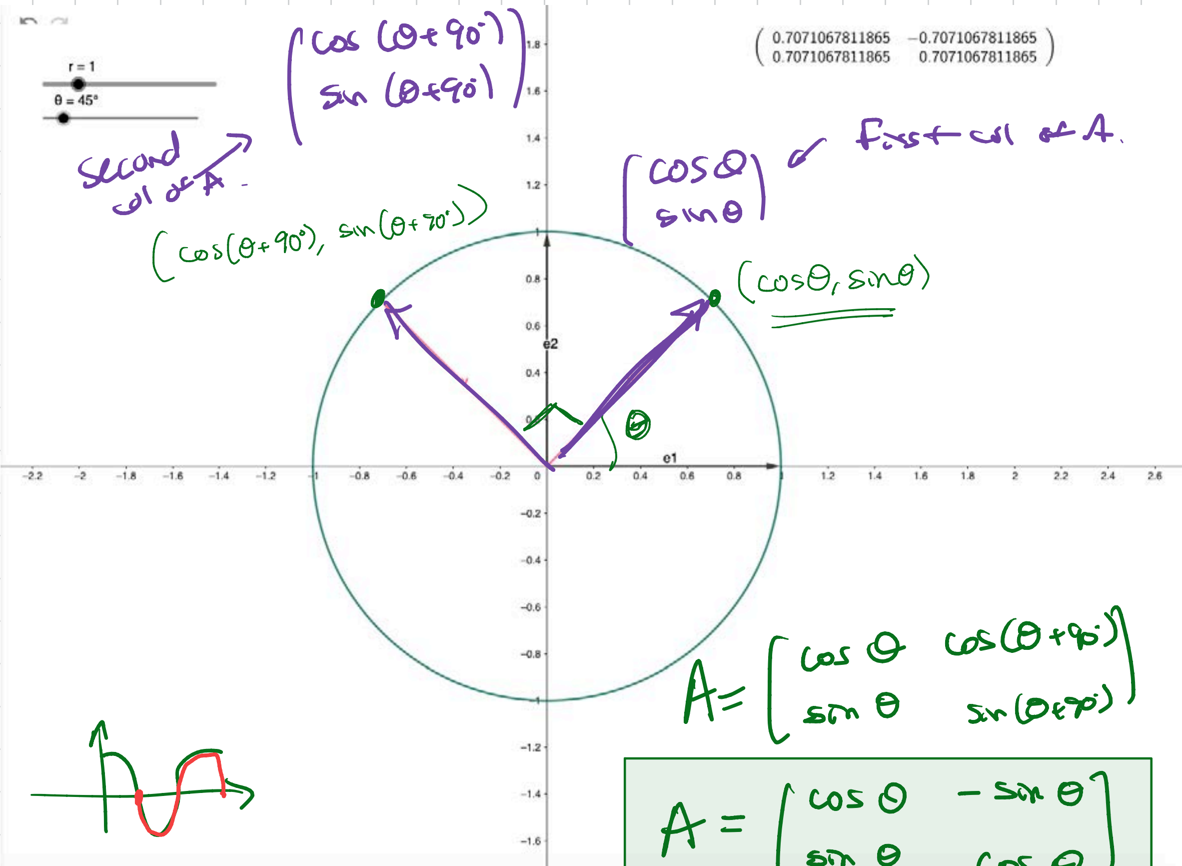
$T(e_1)$ $T(e_2)$

Linear Transformations are Matrix Transformations
Example

Question
What is the matrix for the linear transformation $T: \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$T(x) = x$ rotated counterclockwise by an angle θ?

https://www.geogebra.org/graphing/jyzexjma



$\begin{pmatrix}\cos(\theta+90^\circ)\\ \sin(\theta+90^\circ)\end{pmatrix}$

second col of A

$(\cos(\theta+90^\circ), \sin(\theta+90^\circ))$

$\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$ First col of A.

$(\cos\theta, \sin\theta)$

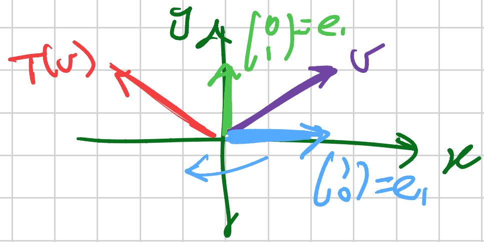
θ

$$A = \begin{bmatrix} \cos\theta & \cos(\theta+90^\circ) \\ \sin\theta & \sin(\theta+90^\circ) \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$$

Ex. Find A for reflect across y-axis

$$A = [T(e_1) \; T(e_2)]$$



$$= \left[T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) \; T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \right]$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Standard matrix for reflect across y-axis.

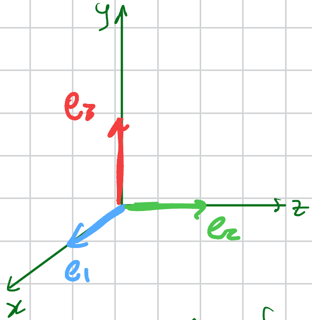
Linear Transformations are Matrix Transformations
Example, continued

Question
What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?

(1) Find the standard matrix A of T

(2) is T one-to-one?

(3) is T onto?



First e_1 reflected to e_1
then e_1 gets projected to $\vec{0}$
So e_1 gets mapped to $\vec{0}$

First e_2 doesn't move when reflected across xy-plane (already in xy-plane)
then doesn't move when projects to yz-plane (already on yz-plane)
e_2 maps to e_2 by T

First e_3 reflects down to $-e_3$
then $-e_3$ doesn't move when projected to yz-plane.
e_3 mapped to $-e_3$ by T

$$A = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) \end{bmatrix} = \boxed{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}$$

Linear Transformations are Matrix Transformations
Example

Question
Define a linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^2$ by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ -y - 5z \end{pmatrix}.$$

What is the standard matrix A for T?

(1) Find the standard matrix A of T — Find $T(e_1), T(e_2)$ & $T(e_3)$

(2) is T one-to-one?

(3) is T onto?

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

First col of A

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

①
$$\begin{array}{cccc} & T(e_1) & T(e_2) & T(e_3) \end{array}$$
$$\boxed{A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \end{bmatrix}}$$

② no b/c not every col of A has a pivot

③ yes b/c every row of A has a pivot

Summary

- **Linear transformations** are the transformations that come from matrices.
- The **unit coordinate vectors** $e_1, e_2, \ldots$ are the unit vectors in the positive direction along the coordinate axes.
- You compute the columns of the matrix for a linear transformation by plugging in the unit coordinate vectors.
- This is useful when the transformation is specified geometrically, in terms of a formula, or any other way that isn't as a matrix transformation.

$\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}$ $n<m$ (tall) $\begin{pmatrix} | & | \\ | & | \\ | & | \end{pmatrix} \sim \begin{pmatrix} | & | \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\begin{bmatrix} * & * \\ * & * \end{bmatrix}$ $n=m$ (square) $\begin{bmatrix} | & | \\ | & | \end{bmatrix} \sim \begin{bmatrix} | & | \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}$ $n>m$ (wide) Not one-to-one!