Section 3.4

Matrix Multiplication

Motivation

Recall: we can turn any system of linear equations into a matrix equation

$$Ax = b.$$

This notation is suggestive. Can we solve the equation by "dividing by A"?

$$\boxed{\cancel{x = \frac{b}{A}}}$$

please don't use notation $\frac{1}{A}$ meaningless.

Answer: Sometimes, but you have to know what you're doing.

Today we'll study *matrix algebra*: adding and multiplying matrices.

These are not so hard to do. The important thing to understand today is the relationship between *matrix multiplication* and *composition of transformations*.

More Notation for Matrices

Let A be an $m \times n$ matrix.

We write a_{ij} for the entry in the ith row and the jth column. It is called the **ijth entry** of the matrix.

$$\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}$$

ith row

jth column

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$a_{11}=1$ $a_{12}=2$ $a_{13}=3$

$\boxed{a_{21}=4}$ $a_{22}=5$ $a_{23}=6$

a_{21} ← column 1.

↑ row 2

a_{ij} ← column j.

↑ row i

The entries $a_{11}, a_{22}, a_{33}, \ldots$ are the **diagonal entries**; they form the **main diagonal** of the matrix.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

A **diagonal matrix** is a *square* matrix whose only nonzero entries are on the main diagonal.

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

The $n \times n$ **identity matrix** I_n is the diagonal matrix with all diagonal entries equal to 1. It is special because $I_n v = v$ for all v in $\mathbf{R}^n$.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

↑ "acting like "1" "

More Notation for Matrices

Continued

The **zero matrix** (of size $m \times n$) is the $m \times n$ matrix 0 with all zero entries.

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose rows are the columns of A. In other words, the ij entry of A^T is a_{ji}.

$$\overset{A}{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}} \rightsquigarrow \overset{A^T}{\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}}$$

flip

a_{12} became a_{21}

$$\underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T} = \underset{3\times 2}{\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}}$$

$$[a_{ij}]^T = [a_{ji}]$$

$$\neq \begin{bmatrix} 4 & 1 \\ 5 & 2 \\ 6 & 3 \end{bmatrix}$$

Matrix Multiplication

Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix with columns $v_1, v_2, \ldots, v_p$: (must be equal)

$$B = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

The **product** AB is the $m \times p$ matrix with columns $Av_1, Av_2, \ldots, Av_p$:

$$AB \overset{\text{def}}{=} \begin{pmatrix} | & | & & | \\ Av_1 & Av_2 & \cdots & Av_p \\ | & | & & | \end{pmatrix}.$$

The equality is a definition

In order for $Av_1, Av_2, \ldots, Av_p$ to make sense, the number of columns of A has to be the same as the number of rows of B. Note the sizes of the product!

Example

$$\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}\begin{pmatrix}1&-3\\2&-2\\3&-1\end{pmatrix} = \left(\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}\cdot\begin{pmatrix}1\\2\\3\end{pmatrix} \quad \begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}\cdot\begin{pmatrix}-3\\-2\\-1\end{pmatrix}\right) = \begin{pmatrix}14&-10\\32&-28\end{pmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad (w_1, w_2, w_3)$$

$$B = \begin{bmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{bmatrix} \quad (v_1, v_2)$$

can compute AB & BA.

$$Av_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{pmatrix}1+4+9\\4+10+18\end{pmatrix} = \begin{pmatrix}14\\32\end{pmatrix} \qquad AB = [Av_1 \ Av_2]$$

$$AB = \begin{bmatrix} 14 & -10 \\ 32 & 28 \end{bmatrix}$$

Bw_1 $\quad Bw_2$ $\quad Bw_3$ $\qquad A = [w_1 \ w_2 \ w_3]$

$$BA = \begin{bmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -11 & -13 & -15 \\ -6 & -6 & -6 \\ -1 & 1 & 3 \end{bmatrix}$$

The ij entry of $C = AB$ is the ith row of A times the jth column of B:
$c_{ij} = (AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$.

$$\begin{pmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mk} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kp} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{pmatrix}$$

ith row $\quad$ jth column $\quad$ ij entry

The Row-Column Rule for Matrix Multiplication

Example

row 1, col 1:
$$\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}\begin{pmatrix}1&-3\\2&-2\\3&-1\end{pmatrix} = \begin{pmatrix}1\cdot1+2\cdot2+3\cdot3 & \square \\ \square & \square\end{pmatrix} = \begin{pmatrix}14 & \square \\ \square & \square\end{pmatrix}$$

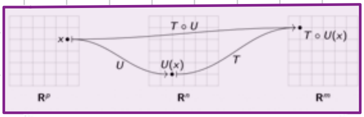
row 2, col 1:
$$\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}\begin{pmatrix}1&-3\\2&-2\\3&-1\end{pmatrix} = \begin{pmatrix}\square & \square \\ 4\cdot1+5\cdot2+6\cdot3 & \square\end{pmatrix} = \begin{pmatrix}\square & \square \\ 32 & \square\end{pmatrix}$$

col 2

$$BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$$

Standard matrix for $T \circ U$ (T after U)

Note: multiplying matrices can help w/ transformations.



The matrix of the composition is the product of the matrices!

Composition of Transformations

Definition

Let $T: \mathbf{R}^n \to \mathbf{R}^m$ and $U: \mathbf{R}^p \to \mathbf{R}^n$ be transformations. The **composition** is the transformation

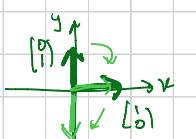
$$T \circ U: \mathbf{R}^p \to \mathbf{R}^m \quad \text{defined by} \quad T \circ U(x) = T(U(x)).$$

composition

$f \circ g(x) = f(g(x))$

"f after g"

Example: Find the standard matrix for the transformation $T \circ U$ where
U is the transformation which rotates vectors in R^2 by 90^deg *clockwise*, and
T is the transformation which projects vectors in R^2 to the x-axis.



$\begin{bmatrix}0\\1\end{bmatrix}$ $\quad \begin{bmatrix}1\\0\end{bmatrix}$

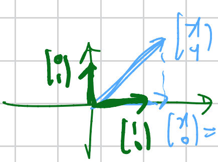
$\begin{bmatrix}0\\-1\end{bmatrix} = U\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$

rotate by 90° clockwise (by itself)

$$A = [U(e_1) \ U(e_2)] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Key idea from section 3.3 about linear transformations

Q: Should we do $\boxed{ABx}$ to get the final vector or $\boxed{BAx}$?? ← first do Ax then do B to that



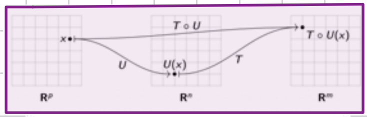
$\begin{bmatrix}0\\1\end{bmatrix}$ $\quad \begin{bmatrix}x\\y\end{bmatrix}$

$\begin{bmatrix}1\\0\end{bmatrix}$ $\quad \begin{bmatrix}x\\0\end{bmatrix} = T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)$

$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix} \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}$

project to x-axis (by itself)

$$B = [T(e_1) \ T(e_2)] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



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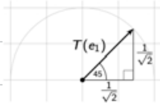
Composition of Linear Transformations
Another Example

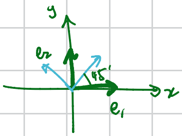
Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be rotation by 45° ccw, and let $U: \mathbf{R}^2 \to \mathbf{R}^2$ scale the x-coordinate by 1.5. Let's compute their standard matrices A and B:

first scale x-coord by 1.5

Also: find the standard matrix for $G = T \circ U$

then rotate by 45° CCW.

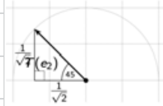




For rotate by 45° (second do rotation T)

$$A = \begin{bmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{bmatrix} = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$$

used formula for rotation by θ

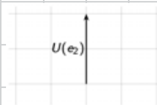


Scale the x-coord by 1.5 (do nothing to y-coord) (first do scaling U)

$U(e_2) = e_2$

$1.5 e_1$ $\begin{bmatrix} 1.5 \\ 0 \end{bmatrix} = U(e_1)$

$$B = [\,U(e_1) \quad U(e_2)\,] = \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}$$



do we want AB or BA?

$$T \circ U(x) = \underbrace{A}_{T \text{ standard matrix}}\ \underbrace{B}_{U \text{ standard matrix}} x$$

$$\text{So } AB = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/\sqrt{2} & \sqrt{2}/\sqrt{2} \end{pmatrix}\begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 3\sqrt{2}/4 & -\sqrt{2}/2 \\ 3\sqrt{2}/4 & \sqrt{2}/2 \end{bmatrix}}$$

✓ standard matrix for $T \circ U$

You Try It!

Poll

Poll
Do there exist *nonzero* matrices A and B with $AB = 0$?

Hint: think geometrically!

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \checkmark$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{\text{project to } x\text{-axis}}\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{project to } y\text{-axis}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \checkmark$$

WARNING!

knowing $A * B = 0$ matrix

does NOT imply

$A = 0$ or $B = 0$

You Try It!

Composition of Linear Transformations
Example

Let $T: \mathbf{R}^3 \to \mathbf{R}^2$ and $U: \mathbf{R}^2 \to \mathbf{R}^3$ be the matrix transformations

$$T(x) = \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}}_{A} x \qquad U(x) = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}}_{B} x.$$

Example: What is the standard matrix for the transformation which first applies the transformation T to a vector x and then applies the transformation U to the result?

$$U(T(x)) = BAx$$

So standard matrix for $U \circ T$ is BA

$$\overset{3\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}} \overset{2\times 3}{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}} = \boxed{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}}$$

standard matrix for $U \circ T$

Addition and Scalar Multiplication for Matrices

You add two matrices component by component, like with vectors.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{pmatrix}$$

Note you can only add two matrices *of the same size.*

You multiply a matrix by a scalar by multiplying each component, like with vectors.

$$c\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \end{pmatrix}.$$

These satisfy the expected rules, like with vectors:

$$A + B = B + A \qquad (A+B)+C = A+(B+C)$$
$$c(A+B) = cA + cB \qquad (c+d)A = cA + dA$$
$$(cd)A = c(dA) \qquad A + 0 = A$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 4 & 4 \end{bmatrix}$$

$$3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}.$$

really do happen

Properties of Matrix Multiplication

Mostly matrix multiplication works like you'd expect. Suppose A has size $m \times n$, and that the other matrices below have the right size to make multiplication work.

$$A(BC) = (AB)C \qquad A(B+C) = (AB + AC)$$
$$(B+C)A = BA + CA \qquad c(AB) = (cA)B$$
$$c(AB) = A(cB) \qquad I_m A = A$$
$$AI_n = A$$

but some you might expect DON'T happen

THESE DON'T WORK!!

Properties of Matrix Multiplication

Caveats

Warnings!

- AB is usually not equal to BA.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$$

not the same!

$AB \neq BA$.

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$$

Warnings!

- $AB = AC$ does not imply $B = C$, even if $A \neq 0$.

$$\overset{A}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}\overset{B}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

So AB same as AC but $B \neq C$.

$$\overset{A}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}\overset{C}{\begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Warnings!

- $AB = 0$ does not imply $A = 0$ or $B = 0$.

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Powers of a Matrix

Definition

Let n be a positive whole number and let A be a square matrix. The **nth power** of A is the product

$$A^n = \underbrace{A \cdot A \cdots A}_{n \text{ times}}$$

Example

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = A^2$$

$$A^3 = \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = A^3$$

Summary

- The product of an $m \times n$ matrix and an $n \times p$ matrix is an $m \times p$ matrix. I showed you two ways of computing the product.
- Composition of linear transformations corresponds to multiplication of matrices.
- You have to be careful when multiplying matrices together, because things like commutativity and cancellation fail.
- You can take powers of square matrices.

$AB \neq BA$

$AB = AC \not\Rightarrow B = C.$

\# cols of A — must match — # rows of B.

$$\underset{m\times n}{A}\ \underset{n\times p}{B} = \underset{m\times p}{C}$$

Section 3.5 and 3.6

Matrix Inverses and the Invertible Matrix Theorem

Compare to numbers.

3 is invertible (using multiplication) b/c $3 * \frac{1}{3} = 1$.

The Definition of Inverse

Definition

Let A be an $n \times n$ square matrix. We say A is **invertible** (or **nonsingular**) if there is a matrix B of the same size, such that

$$\boxed{AB = I_n} \quad \text{and} \quad \boxed{BA = I_n}$$

identity matrix $\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$

In this case, B is the **inverse** of A, and is written A^{-1}.

Example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

I claim $B = A^{-1}$. Check:

$$AB = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{pmatrix} 2-1 & -2+2 \\ 1-1 & -1+2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

$$BA = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2-1 & 1-1 \\ -2+2 & -1+2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

So since $AB = I_2$ and $BA = I_2$ then A and B are inverses $B = A^{-1}$ or $A = B^{-1}$

A^{-1} "A inverse" is the inverse of A.

You Try It!

Poll

Do there exist two matrices A and B such that AB is the identity, but BA is not? If so, find an example. (Both products have to make sense.)

Hint: don't assume A and B have the same size.

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq I_3.$$

However

If A and B are square matrices, then

$$\boxed{AB = I_n} \quad \text{if and only if} \quad \boxed{BA = I_n}.$$

So in this case you only have to check one.

Key observation: Having an inverse of a coefficient matrix of a system Ax=b makes solving the system very easy!

Solving Linear Systems via Inverses

Example

Example

Solve the system

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 &= 1 \\ x_1 + 3x_3 &= 1 \\ 2x_1 + 2x_2 + 3x_3 &= 1 \end{aligned} \quad \text{using} \quad \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}.$$

Idea

Solve $Ax=b$

$A = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Start with

mult. on the left by A^{-1}

$Ax = b \Rightarrow A^{-1}Ax = A^{-1}b$

$\Rightarrow Ix = A^{-1}b$

$\Rightarrow x = A^{-1}b$

to get x just multiply

$A^{-1}b = \begin{bmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Soln.

$\Rightarrow x = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

$2(-2) + 3(1) + 2(1) = 1$ ✓

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 &= 1 \\ \overset{-2}{x_1} \overset{+3}{+ 3x_3} &= 1 \\ 2x_1 + 2x_2 + 3x_3 &= 1 \end{aligned}$$

✓

$-4 + 2 + 3 = 1$ ✓

The advantage of using inverses is it doesn't matter what's on the right-hand side of the = :

$$\begin{cases} 2x_1 + 3x_2 + 2x_3 = b_1 \\ x_1 + 3x_3 = b_2 \\ 2x_1 + 2x_2 + 3x_3 = b_3 \end{cases} \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$= \begin{pmatrix} -6b_1 - 5b_2 + 9b_3 \\ 3b_1 + 2b_2 - 4b_3 \\ 2b_1 + 2b_2 - 3b_3 \end{pmatrix}$$

Computing A^{-1}

The 2×2 case

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The **determinant** of A is the number

$$\boxed{\det(A) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.}$$

Facts:

1. If $\det(A) \neq 0$, then A is invertible and $\boxed{A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$.
2. If $\det(A) = 0$, then A is not invertible.

Example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \frac{1}{(1)(4)-(3)(2)}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Exam 2 on Friday this week.

Come to lecture room IC 211 @ 2pm

(try to come 5-10 min early)

WARNING no phones within reach.

check

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} -2+3 & 1-1 \\ -6+6 & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ yay!

Let T = counterclockwise rotation in the plane by 45°. What is T^{-1}?

Invertible Linear Transformations

Examples

Let T = counterclockwise rotation in the plane by 45°. Its matrix is

$$A = \begin{pmatrix} \cos(45°) & -\sin(45°) \\ \sin(45°) & \cos(45°) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then T^{-1} = counterclockwise rotation by $-45°$. Its matrix is what matrix?

$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

① compute A^{-1}

② check $AA^{-1} = I$.

③ think geometrically.

② $$AA^{-1} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\frac{1}{2} - \left(\frac{-1}{2}\right)}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \frac{1}{1}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

① So $$A^{-1} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Formula rotating clockwise

$$\boxed{A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}$$

Invertible Transformations

Definition

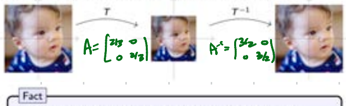
A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^n$ is **invertible** if there exists another transformation $U\colon \mathbf{R}^n \to \mathbf{R}^n$ such that

$$T \circ U(x) = x \quad \text{and} \quad U \circ T(x) = x$$

for all x in $\mathbf{R}^n$. In this case we say U is the **inverse** of T, and we write $U = T^{-1}$.

In other words, $T(U(x)) = x$, so T "undoes" U, and likewise U "undoes" T.

Let T = shrinking by a factor of 2/3 in the plane. What is T^{-1}?



$A = \begin{bmatrix} 2/3 & 0 \\ 0 & 2/3 \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} 3/2 & 0 \\ 0 & 3/2 \end{bmatrix}$

Fact

If T is an invertible linear transformation with matrix A, then T^{-1} is an invertible linear transformation with matrix A^{-1}.

Fact

A transformation T is invertible if and only if it is both one-to-one and onto.

So A has a pivot in every column & every row.

Some Facts

Say A and B are invertible $n \times n$ matrices.

1. A^{-1} is invertible and its inverse is $(A^{-1})^{-1} = A$.
2. AB is invertible and its inverse is $(AB)^{-1} = \cancel{A^{-1}B^{-1}}$ $B^{-1}A^{-1}$.

   Why? $(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$.
3. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

   Why? $A^T(A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n$.

$ABA^{-1}B^{-1}$ doesn't simplify.

lets check

$(AB)^{-1} = B^{-1}A^{-1}$

$A\cancel{B} * \cancel{B^{-1}} A^{-1} \stackrel{?}{=} I$

$= A(I A^{-1})$

$= AA^{-1}$

$= I$ ✓

You Try It!

Question: If A, B, C are invertible $n \times n$ matrices, what is the inverse of ABC?

The inverse of ABC is $C^{-1}B^{-1}A^{-1}$.

lets check $\cancel{A}\cancel{B}\cancel{C} * \cancel{C^{-1}}\cancel{B^{-1}}\cancel{A^{-1}} = I$.

I

I

I.

Computing A^{-1}

In general

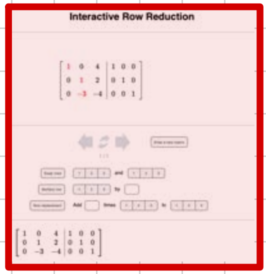
Let A be an $n \times n$ matrix. Here's how to compute A^{-1}.

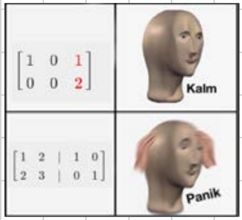
1. Row reduce the augmented matrix $(A \mid I_n)$.
2. If the result has the form $(I_n \mid B)$, then A is invertible and $B = A^{-1}$.
3. Otherwise, A is not invertible.

Example

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}$$

https://textbooks.math.gatech.edu/ila/demos/rrinter.html?mat=1,0,4,1,0,0:0,1,2,0,1,0:0,-3,-4,0,0,1&augment=2





$AA^{-1} = I$?

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{bmatrix}\begin{bmatrix} 1 & -6 & -2 \\ 0 & -2 & -1 \\ 0 & 3/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 3 & 1 \end{array}\right]$$

$$\begin{array}{l} -2R_3+R_1 \\ -R_3+R_2 \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -6 & -2 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 2 & 0 & 3 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \boxed{\begin{matrix} 1 & -6 & -2 \\ 0 & -2 & -1 \\ 0 & 3/2 & 1/2 \end{matrix}} \\ 0 & 0 & 1 & \end{array}\right] = [I \mid A^{-1}]$$

this is A^{-1}

The Invertible Matrix Theorem

A.K.A. The Really Big Theorem of Math 1553

IMT

all $n \times n$ matrices

invertible

not invertible

The Invertible Matrix Theorem

Let A be an $n \times n$ matrix, and let $T: \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. A is invertible.
2. T is invertible.
3. The reduced row echelon form of A is the identity matrix I_n.
4. A has n pivots.
5. $Ax = 0$ has no solutions other than the trivial solution.
6. $\mathrm{Nul}(A) = \{0\}$.
7. $\mathrm{nullity}(A) = 0$.
8. The columns of A are linearly independent.
9. The columns of A form a basis for $\mathbf{R}^n$.
10. T is one-to-one.
11. $Ax = b$ is consistent for all b in $\mathbf{R}^n$.
12. $Ax = b$ has a unique solution for each b in $\mathbf{R}^n$.
13. The columns of A span $\mathbf{R}^n$.
14. $\mathrm{Col}\, A = \mathbf{R}^n$.
15. $\dim \mathrm{Col}\, A = n$.
16. $\mathrm{rank}\, A = n$.
17. T is onto.
18. There exists a matrix B such that $AB = I_n$.
19. There exists a matrix B such that $BA = I_n$.

you really have to know these

The Invertible Matrix Theorem

Summary

There are two kinds of *square* matrices:

1. invertible (non-singular), and
2. non-invertible (singular).

For invertible matrices, all statements of the Invertible Matrix Theorem are true.

For non-invertible matrices, all statements of the Invertible Matrix Theorem are false.

practice the IMT idea by

① Step 1: pick any two items above

② Step 2: try to show they're equivalent.

not common.

terminology: A singular means A NOT invertible

A non-singular means A yes invertible

common

3×3

$$A = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$$

if I pick v_1, v_2, v_3 randomly.

What's the expected rank of A?

usually spans $\mathbb{R}^3$

The Invertible Matrix Theorem
Example

typical. unusual

Question: Is this matrix invertible? Is A non-singular? or is A singular?

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & 7 \\ -2 & -4 & 1 \end{pmatrix}$$

Defn: T is one-to-one if $T(x) = T(y)$ then $x = y$.

(Opposite of one-to-one is many-to-one which means $T(x) = b$ for infinitely many x's)

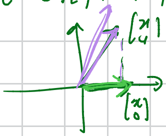
Soln: Let's check if A has 3 pivots (A invertible) or A has fewer than 3 pivots (A singular)

Free column.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 7 \\ -2 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

A has 2 pivots and is NOT invertible (singular A).

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$



$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

You Try It!

The Invertible Matrix Theorem
Another Example

Problem: Let A be a 3×3 matrix such that

$$A\begin{pmatrix} 1 \\ 7 \\ 0 \end{pmatrix} = A\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}.$$

True or False: the rank of A is equal to 3.

Suppose $A\begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix} = b$

So other vector is also b

$A\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = b$

how many solns to $Ax = b$.

The Invertible Matrix Theorem
A.K.A. The Really Big Theorem of Math 1553

The Invertible Matrix Theorem

Let A be an $n \times n$ matrix, and let $T: \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. A is invertible.
2. T is invertible.
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4. A has n pivots.
5. $Ax = 0$ has no solutions other than the trivial solution.
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8. The columns of A are linearly independent.
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10. T is one-to-one.
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13. The columns of A span $\mathbf{R}^n$.
14. $\text{Col}\, A = \mathbf{R}^n$.
15. $\dim \text{Col}\, A = n$.
16. $\text{rank}\, A = n$.
17. T is onto.
18. There exists a matrix B such that $AB = I_n$.
19. There exists a matrix B such that $BA = I_n$.

you really have to know these

these FAIL

So A is not invertible and $\text{rank}\, A \neq 3$.

Summary

- The **inverse** of a square matrix A is a matrix A^{-1} such that $AA^{-1} = I_n$ (equivalently, $A^{-1}A = I_n$).
- If A is invertible, then you can solve $Ax = b$ by "dividing by A": $b = A^{-1}x$. There is a unique solution $x = A^{-1}b$ for every x.
- You compute A^{-1} (and whether A is invertible) by row reducing $\left(A \mid I_n \right)$. There's a trick for computing the inverse of a 2×2 matrix in terms of determinants.
- A linear transformation T is invertible if and only if its matrix A is invertible, in which case A^{-1} is the matrix for T^{-1}.
- The Invertible Matrix theorem is a list of a zillion equivalent conditions for invertibility that you have to learn (and should understand, since it's well within what we've covered in class so far).