Chapter 4

Determinants

Section 4.1

Determinants: Definition

Orientation

Recall: This course is about learning to:

- Solve the matrix equation $Ax = b$
  We've said most of what we'll say about this topic now.
- Solve the matrix equation $Ax = \lambda x$ (eigenvalue problem)
  We are now aiming at this.
- Almost solve the equation $Ax = b$
  This will happen later.

https://strawpoll.com/bVg8BW0m2yY

Today is mostly about the *theory* of the determinant; in the next lecture we will focus on *computation*.

A Definition of Determinant

Definition

The **determinant** is a function — determinants are only for square matrices!

$$\det\colon \{n \times n \text{ matrices}\} \longrightarrow \mathbf{R}$$

with the following properties:

1. If you do a row replacement on a matrix, the determinant doesn't change.
2. If you scale a row by c, the determinant is multiplied by c.
3. If you swap two rows of a matrix, the determinant is multiplied by -1.
4. $\det(I_n) = 1$.

This is a *definition* because it tells you how to compute the determinant: row reduce!

old way: $\det\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} = 2*4 - 1*1 = 8 - 1 = 7$

Example:

$$\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \xrightarrow{③\ R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix} \xrightarrow{①\ -2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 4 \\ 0 & -7 \end{bmatrix} \xrightarrow{②\ -\frac{1}{7}R_2 \to R_2} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \xrightarrow{①\ -4R_2 + R_1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So what's original det??

new det is -1 times old det.

new det same as old det

new det $-\frac{1}{7} *$ old det

$\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$ by rule ④

tracking backwards

undo mult by $-\frac{1}{7}$.

$$\det\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} = 1 * (-7) * 1 * (-1) * 1 = 7$$

Special Cases

Special Case 1
If A has a zero row, then $\det(A) = 0$.

Example:

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix} = 0$$

Special Cases

Special Case 2
If A is upper-triangular, then the determinant is the product of the diagonal entries:

$$\det\begin{pmatrix} a & \star & \star \\ 0 & b & \star \\ 0 & 0 & c \end{pmatrix} = abc.$$

defn upper triangular means 0's below the main diagonal.

(all non-zero entries if any are on or above the diagonal)

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -10 \\ 0 & 0 & -74 \end{pmatrix} = 1 * 1 * (-74) = -74$$

$$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -10 \\ 0 & 0 & 1 \end{pmatrix} \leftarrow \text{has det } 1,$$

new det $-\frac{1}{74} *$ old det

Idea: row reduce to REF (don't need RREF).

Compute the determinant of the matrix.

$$A=\begin{pmatrix} 0 & -7 & -4 \\ 2 & 4 & 6 \\ 3 & 7 & -1 \end{pmatrix} \sim \begin{bmatrix} 2 & 4 & 6 \\ 0 & -7 & -4 \\ 3 & 7 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -7 & -4 \\ 3 & 7 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -7 & -4 \\ 0 & 1 & -10 \end{bmatrix}$$

current row operation → $R_1 \leftrightarrow R_2$

total det calculator → $\det A * -1$

$\frac{1}{2}R_1 \to R_1$

$\det A * -1 * \frac{1}{2}$

$-3R_1 + R_3 \to R_3$

$\det A * -1 * \frac{1}{2} * 1.$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -10 \\ 0 & -7 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -10 \\ 0 & 0 & -74 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$\det A * -1 * \frac{1}{2} * 1 * -1.$

$7R_2 + R_3 \to R_3$

$\det A * \cancel{-1} * \frac{1}{2} * 1 * \cancel{-1} * 1 = -74$

det of this matrix.

So $\det A = -74 * 2 = -148.$

Computing Determinants
2 × 2 Example

Let's compute the determinant of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, a general 2 × 2 matrix.

Case 1

Start w/. if $a \neq 0$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \underset{\frac{1}{a}R_1 + R_2 \to R_2}{\sim} \begin{bmatrix} 1 & b/a \\ c & d \end{bmatrix} \underset{-cR_1 + R_2 \to R_2}{\sim} \begin{bmatrix} 1 & b/a \\ 0 & -cb/a + d \end{bmatrix} = B$$

det is

$\det B = -\frac{cb}{a} + d$

$\det A * \frac{1}{a} = \det B = -\frac{cb}{a} + d$

$\Rightarrow \det A = -cb + da = ad - bc.$

Case 2

if $a = 0$

$$A = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} c & d \\ 0 & b \end{bmatrix}$$

$\det B = cb$

and

$\det A = -cb.$

(Same as $ad - bc$ if $a = 0$)

Formula

$$\boxed{\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc}$$

You try it!

Example: $\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = (1)(4) - (2)(3) = 4 - 6 = \boxed{-2}$

Using formula

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \underset{-3R_1 + R_2 \to R_2}{\sim} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$$

det is -2

same det b/c only did a row replacement (no change to det)

Example: $\det \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 3 & 4 \end{pmatrix}$

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} = B$$

$-3R_2 + R_3 \to R_3$
(no change to det)

$\det B = (3)(1)(-2) = -6$

$\det A = \det B = -6$ b/c only used a row replacements

Poll

Poll
True or false:
(a) Row operations can change the determinant of a matrix. TRUE
(b) Row operations can change whether the determinant of a matrix is equal to zero. FALSE

1. row replace (no change)
2. row scaling by non-zero c (mult. old det by c to get new det)
3. row swaps (mult. old by -1).

Think?

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \overset{???}{\sim} \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} \; ???$$

$0R_2 \to R_2$ (NOT a row operation!)

Determinants and Invertibility

Theorem
A square matrix A is invertible if and only if $\det(A)$ is nonzero.

Why?

- If A is invertible, then its reduced row echelon form is the identity matrix, which has determinant equal to 1.
- If A is not invertible, then its reduced row echelon form has a zero row, hence has zero determinant.
- Doing row operations doesn't change whether the determinant is zero.

IMT DLC

invertible matrix theorem

downloadable content.

Example: Is the matrix invertible? $\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}$

not invertible. b/c (a) $\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}$

so only 1 pivot
1 free var

(b) 2nd col is 1/2 * first col
cols lin dep.

(c) $\det\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix} = 12 - 12 = 0$

So det is 0.

Example: Is the matrix invertible? $\begin{pmatrix} 3 & 2 \\ 6 & 2 \end{pmatrix}$

$\det\begin{pmatrix} 3 & 2 \\ 6 & 2 \end{pmatrix} = 6 - 12 = -6 \neq 0$

So $\begin{pmatrix} 3 & 2 \\ 6 & 2 \end{pmatrix}$ is invertible

Example: Is the matrix invertible? $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$\det\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = 1 - 4 = -3 \neq 0$

so invertible

Determinants and Products

Theorem
If A and B are two $n \times n$ matrices, then

$$\det(AB) = \det(A) \cdot \det(B).$$

Example: Compute the determinant of the matrix. $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 6 & 2 \end{pmatrix} = \begin{bmatrix} 3+12 & 2+4 \\ 6+6 & 4+2 \end{bmatrix} = \begin{bmatrix} 15 & 6 \\ 12 & 6 \end{bmatrix}$

$\det\left(\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 6 & 2 \end{pmatrix}\right) = \det\begin{pmatrix} 15 & 6 \\ 12 & 6 \end{pmatrix} = (15)(6) - (6)(12) = 90 - 72$
$= 18.$

OR:

$\det\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} * \det\begin{pmatrix} 3 & 2 \\ 6 & 2 \end{pmatrix} = (-3)(-6) = 18.$

Example: Compute the determinant of the matrix. $\underset{A}{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}}\underset{B}{\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}} = \begin{bmatrix} 3+12 & 2+8 \\ 6+6 & 4+4 \end{bmatrix} = \begin{bmatrix} 15 & 10 \\ 12 & 8 \end{bmatrix}$

$\det(AB) = \det A * \det B = \det\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} * \det\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix} = (-3)(0)$
$= 0.$

$\det(AB) = \det\begin{pmatrix} 15 & 10 \\ 12 & 8 \end{pmatrix} = 120 - 120 = 0.$ ✓

Transposes
Review

Recall: The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose rows are the columns of A. In other words, the ij entry of A^T is a_{ji}.

$$\underset{A}{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}} \rightsquigarrow \underset{A^T}{\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}}$$

flip

Determinants and Transposes

Theorem
If A is a square matrix, then

$$\det(A) = \det(A^T),$$

where A^T is the transpose of A.

2x2 3x3, 4x4, ... nxn etc.

Example: $\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \det\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\det A = 4 - 6 = -2$

$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ $\det A^T = 4 - 6 = -2.$

As a consequence, det behaves the same way with respect to *column* operations as row operations.

Corollary ← an immediate consequence of a theorem
If A has a zero column, then $\det(A) = 0$.

Corollary
The determinant of a *lower-triangular* matrix is the product of the diagonal entries.

(The transpose of a lower-triangular matrix is upper-triangular.)

Example: Compute the determinant of the matrix. $\begin{pmatrix} 1 & 0 & 3 \\ 2 & 0 & 2 \\ 6 & 0 & 4 \end{pmatrix}$

b/c det A = det A^T

b/c row of zeros means det is zero.

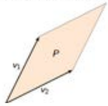
$$\det\begin{pmatrix} 1 & 0 & 3 \\ 2 & 0 & 2 \\ 6 & 0 & 4 \end{pmatrix} = \det\begin{pmatrix} 1 & 2 & 6 \\ 0 & 0 & 0 \\ 3 & 2 & 4 \end{pmatrix} = 0$$

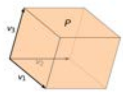
Section 4.3

Determinants and Volumes

Determinants and Volumes

The columns $v_1, v_2, \ldots, v_n$ of an $n \times n$ matrix A give you n vectors in $\mathbf{R}^n$. These determine a **parallelepiped** P.





Theorem

Let A be an $n \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, and let P be the parallelepiped determined by A. Then

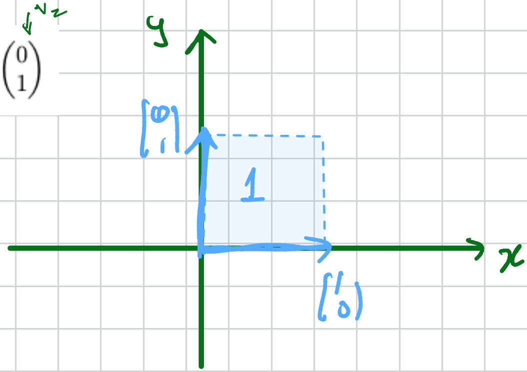
$$\boxed{(\text{volume of } P) = |\det(A)|.}$$

don't forget $|\det A|$ abs value.

n- dimensional volume
length if $n=1$
area if $n=2$.
volume if $n \geq 3$.

Example: Find the volume of the parallelpiped determined by the vectors.

(a) $\overset{v_1}{\begin{pmatrix}1\\0\end{pmatrix}}, \overset{v_2}{\begin{pmatrix}0\\1\end{pmatrix}}$



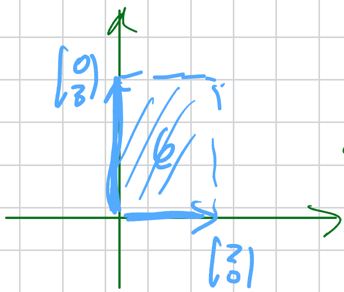
$$A = \begin{array}{cc} v_1 & v_2 \end{array}$$
$$A = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$\det A = 1 - 0 = 1$. ✓

new $A = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

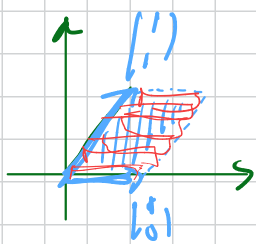
$\det A = 0 - 1 = -1$.

(b) $\begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}0\\3\end{pmatrix}$



$\det \begin{bmatrix}2 & 0\\0 & 3\end{bmatrix} = 6.$

(c) $\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}$



$\det \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix} = 0 - 1 = -1$

So area is $\left|\det \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\right| = 1$

(d) $\begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\3\end{pmatrix}$

Section 4.3

Determinants and Volumes

Determinants and Volumes

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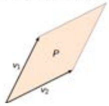
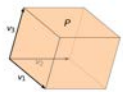




Theorem

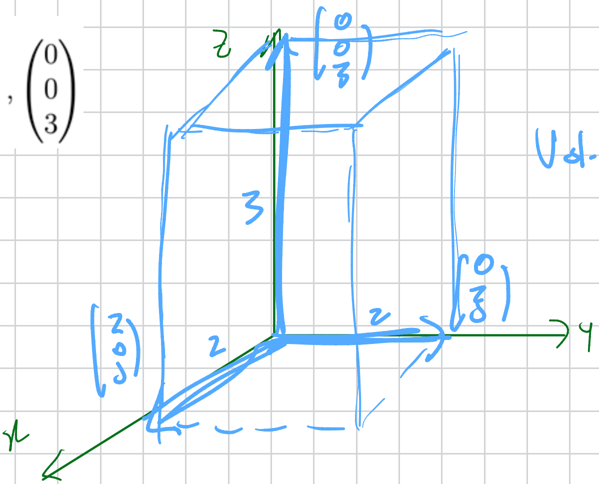
Let A be an $n \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, and let P be the parallelepiped determined by A. Then

$$(\text{volume of } P) = |\det(A)|.$$ ⟵ don't forget $|\det A|$ abs value   n- di   volume

Example: Find the volume of the parallelpiped determined by the vectors.

v_1

(d) $\begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\3\end{pmatrix}$



Vol. is $2*2*3 = 12$

$\det\begin{bmatrix}2&0&0\\0&2&0\\0&0&3\end{bmatrix} = 2*2*3$

$= 12.$

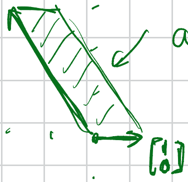
Sanity check: the volume of P is zero $\iff$ the columns are *linearly dependent*
(P is "flat") $\iff$ the matrix A is not invertible.

Determinants and Volumes
Examples in $\mathbb{R}^2$

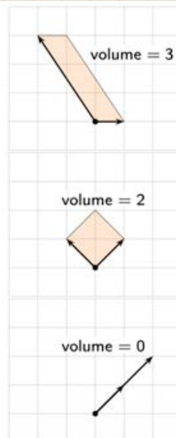
You try it!

Find (a) the determinant of the given matrix, (b) sketch the parallelpiped determined by the columns of the matrix, and (c) determine the volume of the parallelpiped.

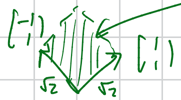
$$\det\begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} = (1)(3) - (-2)(0) = 3$$



area is 3



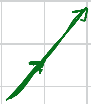
$$\det\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = -1-1 = -2$$



area 2

Warning!
area is not -2

$$\det\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = 2-2 = 0$$



no area
(area equals 0)

Determinants and Volumes

Theorem
Let A be an $n \times n$ matrix, and let $T(x) = Ax$. If S is any region in $\mathbf{R}^n$, then

$$(\text{volume of } T(S)) = |\det(A)|\,(\text{volume of } S).$$

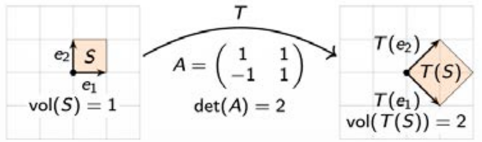
Example: Let S be the unit disk in $\mathbf{R}^2$, and let $T(x) = Ax$ for

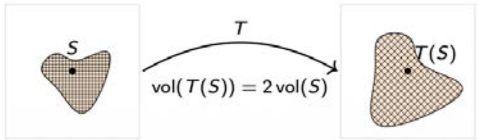
area is π

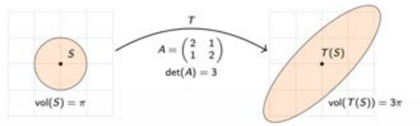
$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \qquad \det A = 4 - 1 = 3$$

Area of new shape $= |\det A| *$ area of old shape

$$\boxed{\text{new area} = 3 * \pi}$$







Summary

Magical Properties of the Determinant

you really have to know these

1. There is one and only one function $\det: \{\text{square matrices}\} \to \mathbf{R}$ satisfying the properties (1)–(4) on the second slide.
2. $\boxed{A \text{ is invertible if and only if } \det(A) \neq 0.}$ IMT DLC.
3. The determinant of an upper- or lower-triangular matrix is the product of the diagonal entries.
4. If we row reduce A to row echelon form B using r swaps, then
$$\det(A) = (-1)^r \frac{(\text{product of the diagonal entries of } B)}{(\text{product of the scaling factors})}.$$
5. $\det(AB) = \det(A)\det(B)$ and $\det(A^{-1}) = \det(A)^{-1}$.
6. $\det(A) = \det(A^T)$.
7. $|\det(A)|$ is the volume of the parallelepiped defined by the columns of A.
8. If A is an $n \times n$ matrix with transformation $T(x) = Ax$, and S is a subset of $\mathbf{R}^n$, then the volume of $T(S)$ is $|\det(A)|$ times the volume of S. (Even for curvy shapes S.)

Example

e.g.

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \boxed{\det A = -2}$$

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$\det A^{-1} = (-1/2)(1/2) - (1/2)(1/2)$$
$$= -\frac{1}{4} - \frac{1}{4} = -1/2$$

$$\boxed{\det A^{-1} = \frac{-1}{2}}$$

Proof

$$\det A^{-1} = \frac{1}{\det A}$$

why? $A * A^{-1} = I$

$$\det(A) * \det(A^{-1}) = \det I$$
$$\det A * \det(A^{-1}) = 1$$
$$\Rightarrow \boxed{\det A^{-1} = \frac{1}{\det A}} \checkmark$$

Section 4.2

Cofactor Expansions

Orientation

Last time: we learned...

- ...the definition of the determinant.
- ...to compute the determinant using row reduction.
- ...all sorts of magical properties of the determinant, like
  - $\det(AB) = \det(A)\det(B)$
  - the determinant computes volumes
  - nonzero determinants characterize invertability
  - etc.

Today: we will learn...

- Special formulas for 2×2 and 3×3 matrices.
- How to compute determinants using *cofactor expansions*.
- ~~How to compute inverses using determinants.~~

Determinants of 2×2 Matrices

Reminder

We already have a formula in the 2×2 case:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Determinants of 3×3 Matrices

Here's the formula:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{array}{l} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ \quad - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \end{array}$$

$$a_{11}C_{11} - a_{12}C_{12} + a_{13}C_{13}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Cofactor Expansions

Example

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\det\begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix} = 5\det\begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix} - 1\det\begin{pmatrix} -1 & 2 \\ 4 & -1 \end{pmatrix} + 0\det\begin{pmatrix} -1 & 3 \\ 4 & 0 \end{pmatrix}$$

$$= 5\big((3)(-1) - (2)(0)\big) - 1\big((-1)(-1) - (2)(4)\big) + 0\big((-1)(0) - (3)(4)\big)$$

$$= 5(-3) - (-7) + 0 = -15 + 7 = \boxed{-8}$$

Signs use this pattern $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

expand along row i (odd i)

$\det(A) = a_{i1}C_{i1} \cancel{+} a_{i2}C_{i2} + a_{i3}C_{i3}$

expand down column j (odd j).

$\det(A) = a_{1j}c_{1j} \cancel{+} a_{2j}C_{2j} + a_{3j}C_{3j}$

Cofactor Expansion

Example

Compute det(A), and notice the last column of mostly zeros.

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 5 & 9 & 1 \end{pmatrix}$$

$$\det A = 0 \det\begin{pmatrix} 1 & 1 \\ 5 & 9 \end{pmatrix} - 0 \det\begin{pmatrix} 2 & 1 \\ 5 & 9 \end{pmatrix} + 1 \det\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= 0 - 0 + (2-1) = 1.$$

You Try It!

Poll

Poll

$$\det\begin{pmatrix} 0 & 7 & 2 & 9 & 8 \\ 1 & 3 & 2 & 7 & 4 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 5 \end{pmatrix} = ?$$

A. −84 B. −28 C. −7 D. 0 E. 7 F. 28 G. 84

$$\begin{pmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{pmatrix}$$

using row 3

opt 1

$$\det A = 3 \det\begin{bmatrix} 0 & 7 & 2 & 9 \\ 1 & 3 & 2 & 7 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} = -3 \det\begin{bmatrix} 1 & 3 & 2 & 7 \\ 0 & 7 & 2 & 9 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} = -3*1*7*2*2$$

$$= -4*21$$

$$= -84$$

using column 1.

opt 2

$$\det A = -1 \det\begin{bmatrix} 7 & 2 & 9 & 8 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 5 \end{bmatrix} = + \det\begin{bmatrix} 7 & 2 & 9 & 8 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 5 \end{bmatrix}$$

$$= -\det\begin{bmatrix} 7 & 2 & 9 & 8 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 3 \end{bmatrix} = -7*4*3$$

$$= -84$$

Cofactor Expansion

Advice

- In general, computing a determinant by cofactor expansion is slower than by row reduction.
- It makes sense to expand by cofactors if you have a row or column with a lot of zeros.
- Also if your matrix has unknowns in it, since those are hard to row reduce (you don't know where the pivots are).

You can also use more than one method; for example:

- Use cofactors on a 4×4 matrix but compute the minors using the 3×3 formula.
- Do row operations to produce a row/column with lots of zeros, then expand cofactors (but keep track of how you changed the determinant!).

Example:

$$\det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

$$\begin{bmatrix} + & \boxed{-} & + \\ - & \boxed{+} & - \\ + & \boxed{-} & + \end{bmatrix}$$

Opt 1 using row 1.

$$\det A = +5 \det\begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix} - 1 \det\begin{pmatrix} -1 & 2 \\ 4 & -1 \end{pmatrix} + 0$$

$$= 5(-3) - (1-7) = -15+7 = -8.$$

Opt 2 using col 2.

$$\det A = -1 \det\begin{pmatrix} -1 & 2 \\ 4 & -1 \end{pmatrix} + 3 \det\begin{pmatrix} 5 & 0 \\ 4 & -1 \end{pmatrix} - 0$$

$$= -(1-8) + 3(-5) = 7 - 15 = -8$$

Summary

We have several ways to compute the determinant of a matrix.

- Special formulas for 2×2 and 3×3 matrices.

  These work great for small matrices.
- Cofactor expansion.

  This is perfect when there is a row or column with a lot of zeros, or if your matrix has unknowns in it.
- Row reduction.

  This is the way to go when you have a big matrix which doesn't have a row or column with a lot of zeros.
- Any combination of the above.

  Cofactor expansion is recursive, but you don't have to use cofactor expansion to compute the determinants of the minors! Or you can do row operations and then a cofactor expansion.