Chapter 5

Eigenvalues and Eigenvectors

Section 5.1

Eigenvalues and Eigenvectors

Definition
If A is an $n \times n$ square matrix and there is a nonzero vector v in $\mathbf{R}^n$ and a scalar λ in $\mathbf{R}$ with

$$Av = \lambda v,$$

then v is an **eigenvector** of A with **eigenvalue** λ.

If Av is a scalar mult. of v. ($\vec{v} \neq \vec{0}$)
then v is an eigenvector of A.

Note: Eigenvectors are by definition nonzero. Eigenvalues may be equal to zero.

Note: This is the most important definition in the course.

Verifying Eigenvectors

Example

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$$

Q: Is v an eigenvector of A?

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} = \begin{bmatrix} 0+24+8 \\ 8+0+0 \\ 0+2+0 \end{bmatrix} = \begin{bmatrix} 32 \\ 8 \\ 2 \end{bmatrix}$$

is this a scalar mult of $v = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$?

$$= 2\begin{bmatrix} 16 \\ 4 \\ 1 \end{bmatrix}$$

So $\lambda = 2$.

So $v = \begin{bmatrix} 16 \\ 4 \\ 1 \end{bmatrix}$ is an eigenvector of A w/ eigenvalue $\lambda = 2$

Example

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$Av = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \overset{?}{=} \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

yes $\lambda = 4$ is the eigenvalue for the eigenvector $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$Av = \lambda v$ is true for any λ if $v = 0$

You Try It!

Poll

Poll
Which of the vectors
A. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ B. $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ C. $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ D. $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ E. $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
are eigenvectors of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$?
What are the eigenvalues?

A. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $\lambda = 2$ yes!

B. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\lambda = 0$ yes!

$= 0\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$Av = 0v$ is true!

C. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\lambda = 0$ yes.

D: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ No $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ not scalar mult of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

E. NO

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\cancel{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$= 0\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$(\lambda = 0)$ ✓

$= 5\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda = 5$?

note If $Av = \lambda v$
then $A(cv) = cAv$
$= c\lambda v$
$= \lambda(cv)$

then cv is also an eigenvector w/ the SAME λ eigenvalue.

Verifying Eigenvalues

Question: Is $\lambda = 3$ an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$?

Q: is there a **nonzero** vector x such that Ax=3x?

any choice of x_2 solves this system e.g. $x_2 = 1$

Q: Can you solve $\boxed{Ax = 3x}$ for x?

$$X = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

$$\begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \end{bmatrix}$$

Solve?

Check:

$$\begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}\begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -8-4 \\ 4-1 \end{pmatrix} = \begin{pmatrix} -12 \\ 3 \end{pmatrix} = 3\begin{pmatrix} -4 \\ 1 \end{pmatrix} \checkmark$$

$$\Rightarrow \begin{cases} 2x - 4y = 3x \\ -x - y = 3y \end{cases}$$

$$A\begin{bmatrix} x \\ y \end{bmatrix} = 3\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow A\left(\begin{bmatrix} x \\ y \end{bmatrix} - 3\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow A\left(\begin{bmatrix} x \\ y \end{bmatrix} - 3I\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow (A - 3I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

parametric vector form

$$\Rightarrow \begin{cases} -x - 4y = 0 \\ -x - 4y = 0 \end{cases} \Rightarrow \begin{bmatrix} -1 & -4 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solve by row reduction

$$A - 3I = \begin{bmatrix} -1 & -4 \\ -1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix} \qquad x = x_2\begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

x_2

https://textbooks.math.gatech.edu/ila/demos/rabbits/book_demo.html

A Biology Question
Motivation

In a population of rabbits:

1. half of the newborn rabbits survive their first year;
2. of those, half survive their second year;
3. their maximum life span is three years;
4. rabbits have 0, 6, 8 baby rabbits in their three years, respectively.

If you know the population one year, what is the population the next year?

$$f_n = \text{first-year rabbits in year } n$$
$$s_n = \text{second-year rabbits in year } n$$
$$t_n = \text{third-year rabbits in year } n$$

The rules say:

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ s_{n+1} \\ t_{n+1} \end{pmatrix}.$$

Let $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$ and $v_n = \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix}$. Then $\boxed{Av_n = v_{n+1}.}$ ⟵ difference equation

A Biology Question
Continued

If you know v_0, what is v_{10}?

$$v_{10} = Av_9 = AAv_8 = \cdots = A^{10}v_0.$$

This makes it easy to compute examples by computer: [interactive]

v_0	v_{10}	v_{11}
$\begin{pmatrix} 3 \\ 7 \\ 9 \end{pmatrix}$	$\begin{pmatrix} 30189 \\ 7761 \\ 1844 \end{pmatrix}$	$\begin{pmatrix} 61316 \\ 15095 \\ 3881 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 9459 \\ 2434 \\ 577 \end{pmatrix}$	$\begin{pmatrix} 19222 \\ 4729 \\ 1217 \end{pmatrix}$
$\begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix}$	$\begin{pmatrix} 28856 \\ 7405 \\ 1765 \end{pmatrix}$	$\begin{pmatrix} 58550 \\ 14428 \\ 3703 \end{pmatrix}$

What do you notice about these numbers?

1. Eventually, each segment of the population doubles every year: $Av_n = v_{n+1} = 2v_n$.
2. The ratios get close to $(16 : 4 : 1)$:

$$v_n = (\text{scalar}) \cdot \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}.$$

Problem: One of the eigenvalues of A is $\lambda = 8$. Identify the other eigenvalue and find at least two eigenvectors for each eigenvalue for the matrix $A = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$

Important

If x is a nonzero vector in Nul(A), then x is an eigenvector of A with eigenvalue $\lambda = 0$.

Consequently, non-invertible matrices always have $\lambda = 0$ as an eigenvalue.

Q: A is invertible or singular?

e.g. $\det A = 2*6 - 4*3 = 0$

Since $\det A = 0 \Rightarrow A$ not invertible.

So $\lambda = 0$ is an eigenvalue.

and an eigenvector is found by row reducing & doing parametric vector form,

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \qquad x = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

So $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector w/ $\lambda = 0$

also $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$, or $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ also eigenvectors w/ $\lambda = 0$.

why? Suppose x in Nul A, so

$$Ax = \vec{0},$$

$$\Rightarrow Ax = 0\vec{x}.$$

$\uparrow \lambda = 0.$

Check $\lambda = 0$

$$\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4+4 \\ -6+6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A\begin{bmatrix} -2 \\ 1 \end{bmatrix} = 0\begin{bmatrix} -2 \\ 1 \end{bmatrix} \checkmark$$

Q: Find eigenvectors (at least 2) for $\lambda = 8$.

Solve $Ax = 8x$

$\Rightarrow Ax - 8x = 0$

$\Rightarrow (A - 8I)x = 0$

Use this one. & find null space

$$A - 8I = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} -6 & 4 \\ 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2/3 \\ 0 & 0 \end{bmatrix}$$

$$x = x_2 \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}.$$

So e.g. an eigenvector $x = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$ w/ eigenvalue $\lambda = 8$.

Check $\lambda = 8$

$$Ax = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 2/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/3 + 4 \\ 2 + 6 \end{bmatrix} = \begin{bmatrix} 16/3 \\ 8 \end{bmatrix} = 8\begin{bmatrix} 2/3 \\ 1 \end{bmatrix} \checkmark$$

Idea: to get other eigenvector mult x by 3 to get $3x$.

new eigenvector is $3x = 3\begin{bmatrix} 2/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

w/ the eigenvalue $\lambda = 8$.

$$\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4+12 \\ 6+18 \end{bmatrix} = \begin{bmatrix} 16 \\ 24 \end{bmatrix} = 8\begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Warning! T F If x eigenvector of A w/ $\lambda = 8$ then $3x$ eigenvector of A w/ $\lambda = 24$

$$A(3x) = 3Ax = 3 \cdot 8x = 8(3x)$$

Eigenspaces

Definition
Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The λ-**eigenspace** of A is the set of all eigenvectors of A with eigenvalue λ, plus the zero vector:

$$\begin{aligned}\lambda\text{-eigenspace} &= \{v \text{ in } \mathbf{R}^n \mid Av = \lambda v\} \\ &= \{v \text{ in } \mathbf{R}^n \mid (A - \lambda I)v = 0\} \\ &= \mathrm{Nul}(A - \lambda I).\end{aligned}$$

Since the λ-eigenspace is a null space, it is a *subspace* of $\mathbf{R}^n$.

How do you find a basis for the λ-eigenspace? Parametric vector form!

Eigenspaces
Example

Find a basis for the 3-eigenspace of

$$A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}.$$

Solve $(A-3I)x=0$ & get parametric vector form

$$A-3I = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ -1 & -4 \end{bmatrix} \sim \begin{array}{cc} & x_2 \\ \left[\begin{matrix} 1 \\ 0 \end{matrix}\right. & \left.\begin{matrix} 4 \\ 0 \end{matrix}\right] \end{array}$$

$$x = x_2 \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

3-eigenspace is $\mathrm{Span}\left\{\begin{bmatrix} -4 \\ 1 \end{bmatrix}\right\} = \mathrm{Nul}(A-3I)$

so a basis is $\boxed{\left\{\begin{bmatrix} -4 \\ 1 \end{bmatrix}\right\}}$

Eigenspaces
Example

Find a basis for the 2-eigenspace of

$$A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}.$$

META:
1. Compute $A-\lambda I$
2. row reduce to RREF & get parametric vector form
3. extract the vectors to get a basis for $\mathrm{Nul}(A-\lambda I)$.

1

$$A-2I = \begin{bmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3/2 & 0 & 3 \\ -3/2 & 0 & -3 \\ -3/2 & 0 & -3 \end{bmatrix} \sim \begin{array}{ccc} & x_2 & x_3 \\ \left[\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} & \left.\begin{matrix} 2 \\ 0 \\ 0 \end{matrix}\right] \end{array} \quad \begin{cases} x_1 + 0x_2 + 2x_3 = 0 \\ x_2 = x_2 \text{ free} \\ x_3 = x_3 \text{ free} \end{cases}$$

$$\begin{cases} x_1 = -2x_3 \\ x_2 = x_2 \text{ free} \\ x_3 = x_3 \text{ free} \end{cases} \qquad x = \begin{bmatrix} -2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

So basis for 2-eigenspace $\boxed{\left\{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}}$

& 2-eigenspace is $\mathrm{Span}\left\{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\} = \mathrm{Nul}(A-2I)$

Question: how many eigenvalues can a matrix have? Can they have more than one eigenvalue?

Eigenspaces
Example

Find a basis for the $\frac{1}{2}$-eigenspace of

$$A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}.$$

(the 1/2-eigenspace must be a line due to theory reasons)

$$A - \frac{1}{2}I = \begin{bmatrix} 7/2 - 1/2 & 0 & 3 \\ -3/2 & 2 - 1/2 & -3 \\ -3/2 & 0 & -1 - 1/2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ -3/2 & 3/2 & -3 \\ -3/2 & 0 & -3/2 \end{bmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ -3 & 3 & -6 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{array}{ccc} & & \checkmark x_2 \\ \hline \end{array} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$X = x_2 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$ so basis for 1/2 - eigenspace is

$$\boxed{\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}}$$

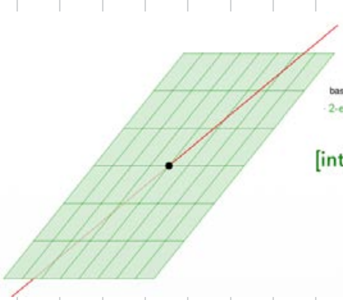
and $\frac{1}{2}$ - eigenspace is the line $\text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

https://textbooks.math.gatech.edu/ila/demos/eigenspace.html?nomult

Eigenspaces
Example: picture

$$A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}.$$

We computed bases for the 2-eigenspace and the 1/2-eigenspace:



basis for the $\frac{1}{2}$-eigenspace: $\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$

basis for the 2-eigenspace: $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$

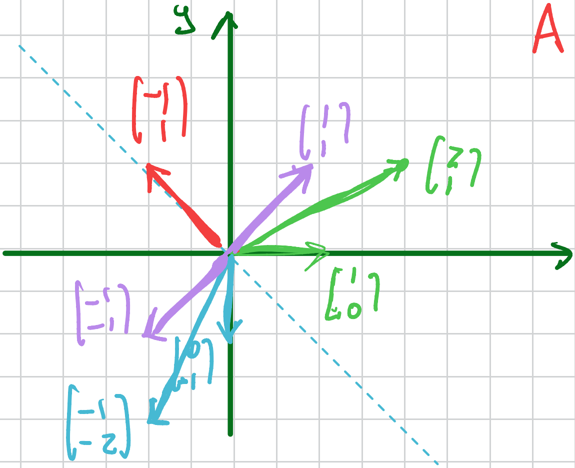
[interactive]

Eigenspaces
Geometry; example

Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!

https://textbooks.math.gatech.edu/ila/demos/eigenspace.html?mat=0,-1:-1,0&nomult



$A\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$

So $\lambda = 1$.

$A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-1\\-1\end{bmatrix}$

So $\lambda = -1$.

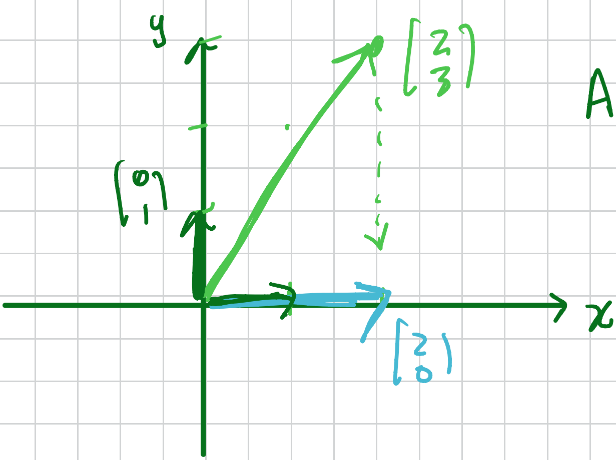
here $A = \begin{pmatrix} \overset{T(e_1)}{0} & \overset{T(e_2)}{-1} \\ -1 & 0 \end{pmatrix}$

Eigenspaces
Geometry; example

Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be the vertical projection onto the x-axis, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!

https://textbooks.math.gatech.edu/ila/demos/eigenspace.html?mat=1,0:0,0&nomult



$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$ $\lambda = 1$.

$A\begin{bmatrix}0\\1\end{bmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ $\lambda = 0$.

A Matrix is Invertible if and only if Zero is not an Eigenvalue

Fact: A is invertible if and only if 0 is not an eigenvalue of A.

Why?

$$\begin{aligned} 0 \text{ is an eigenvalue of } A &\iff Ax = 0x \text{ has a nontrivial solution} \\ &\iff Ax = 0 \text{ has a nontrivial solution} \\ &\iff A \text{ is not invertible.} \end{aligned}$$

(defn of eigenvector)

(IMT).

invertible matrix theorem

DLC.

The Invertible Matrix Theorem
Addenda

We have a couple of new ways of saying "A is invertible" now:

The Invertible Matrix Theorem
Let A be a square $n \times n$ matrix, and let $T: \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. A is invertible.
2. T is invertible.
3. The reduced row echelon form of A is I_n.
4. A has n pivots.
5. $Ax = 0$ has no solutions other than the trivial one.
6. $\mathrm{Nul}(A) = \{0\}$.
7. $\mathrm{nullity}(A) = 0$.
8. The columns of A are linearly independent.
9. The columns of A form a basis for $\mathbf{R}^n$.
10. T is one-to-one.
11. $Ax = b$ is consistent for all b in $\mathbf{R}^n$.
12. $Ax = b$ has a unique solution for each b in $\mathbf{R}^n$.
13. The columns of A span $\mathbf{R}^n$.
14. $\mathrm{Col}\,A = \mathbf{R}^n$.
15. $\dim \mathrm{Col}\,A = n$.
16. $\mathrm{rank}\,A = n$.
17. T is onto.
18. There exists a matrix B such that $AB = I_n$.
19. There exists a matrix B such that $BA = I_n$.
20. The determinant of A is *not* equal to zero.
21. The number 0 is *not* an eigenvalue of A.

Eigenvectors with Distinct Eigenvalues are Linearly Independent

Fact: If $v_1, v_2, \ldots, v_k$ are eigenvectors of A with *distinct* eigenvalues $\lambda_1, \ldots, \lambda_k$, then $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

Warning! Knowing vectors v1 and v2 have the **same** eigenvalue does NOT mean that they are linearly dependent!

You Try It!

Poll

Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be counterclockwise rotation by 45°, and let A be the matrix for T.

Poll
Find an eigenvector of A without doing any computations.
A. Okay. **B.** No way.

Warning!
λ could be complex!! (not until §5.5)

Eigenspaces
Summary

Let A be an $n \times n$ matrix and let λ be a number.

1. λ is an eigenvalue of A if and only if $(A - \lambda I)x = 0$ has a nontrivial solution, if and only if $\mathrm{Nul}(A - \lambda I) \neq \{0\}$.
2. In this case, finding a basis for the λ-eigenspace of A means finding a basis for $\mathrm{Nul}(A - \lambda I)$ as usual, i.e. by finding the parametric vector form for the general solution to $(A - \lambda I)x = 0$.
3. The eigenvectors with eigenvalue λ are the nonzero elements of $\mathrm{Nul}(A - \lambda I)$, i.e. the nontrivial solutions to $(A - \lambda I)x = 0$.

Summary

- **Eigenvectors** and **eigenvalues** are the most important concepts in this course.
- Eigenvectors are by definition nonzero; eigenvalues may be zero.
- The eigenvalues of a triangular matrix are the diagonal entries. next section.
- A matrix is invertible if and only if zero is not an eigenvalue.
- Eigenvectors with distinct eigenvalues are linearly independent.
- The λ-eigenspace is the set of all λ-eigenvectors, plus the zero vector.
- You can compute a basis for the λ-eigenspace by finding the parametric vector form of the solutions of $(A - \lambda I_n)x = 0$.

Q: How to find the λ's?

Section 5.2

The Characteristic Polynomial

The Characteristic Polynomial

Definition

Let A be a square matrix. The **characteristic polynomial** of A is

$$f(\lambda) = \det(A - \lambda I).$$

The **characteristic equation** of A is the equation

$$f(\lambda) = \det(A - \lambda I) = 0.$$

Important

The eigenvalues of A are the roots of the characteristic polynomial $f(\lambda) = \det(A - \lambda I)$.

Key observation

$\boxed{(A - \lambda I)x = 0 \text{ has a nonzero soln for } x.}$

$\iff A - \lambda I$ not invertible

$\iff \boxed{\det(A - \lambda I) = 0}$

The Characteristic Polynomial

Example

Question: What are the eigenvalues of

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}?$$

Step 1: compute $\det(A - \lambda I) = p(\lambda)$ the characteristic polynomial.

Step 2: set $p(\lambda) = 0$ & solve for λ

$$p(\lambda) = \det\begin{pmatrix} 5-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = (5-\lambda)(1-\lambda) - (2)(2)$$

$$= 5 - \lambda - 5\lambda + \lambda^2 - 4$$

$$= \lambda^2 - 6\lambda + 1 = 0$$

Solve using

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So

$$\lambda = \frac{6 \pm \sqrt{36 - 4(1)(1)}}{2}$$

$$\lambda = 3 \pm \frac{\sqrt{32}}{2} = 3 \pm \frac{\sqrt{16 \cdot 2}}{2}$$

$$= 3 \pm \frac{4\sqrt{2}}{2}$$

$$\boxed{\lambda = 3 \pm 2\sqrt{2}}$$

You Try It!

Q: What are the eigenvalues of A?

$$A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$$

Step 1: compute determinant of

$$A - \lambda I = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & -4 \\ -1 & -1-\lambda \end{bmatrix}.$$

$$\det\begin{bmatrix} 2-\lambda & -4 \\ -1 & -1-\lambda \end{bmatrix} = (2-\lambda)(-1-\lambda) - 4$$

$$= (\lambda - 2)(\lambda + 1) - 4$$

$$= \lambda^2 - 2\lambda + \lambda - 2 - 4$$

$$\boxed{= \lambda^2 - \lambda - 6} = (\lambda - 3)(\lambda + 2) = 0$$

So $\boxed{\lambda = 3, -2}$

trick: $p(\lambda) = \lambda^2 - 1\lambda + (-6)$

trace $A = 2 - 1 = 1$

det $A = -2 - 6 = -6$

Problem: Find the characteristic polynomial of A and then find the eigenvalues of the matrix where $A = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix}$

Recall from before

If A is upper or lower triangular, then the determinant of A is the product of the diagonal entries.

Consequently, if A is upper or lower triangular, then the eigenvalues of A are just the diagonal entries.

Warning! This trick does **not work** to find the eigenvalues unless A is upper/lower triangular!

Step 1: compute $\det(A - \lambda I)$

$$= \det\begin{pmatrix} 3-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ 2 & 1 & 1-\lambda \end{pmatrix}$$

$$= (3-\lambda)\det\begin{pmatrix} 2-\lambda & 0 \\ 1 & 1-\lambda \end{pmatrix} - 0 + 0$$

$$= (3-\lambda)\left[(2-\lambda)(1-\lambda) - 0\right]$$

$$= (3-\lambda)(2-\lambda)(1-\lambda) = 0$$

So $\boxed{\lambda = 3, 2, 1}$

The Characteristic Polynomial
Example

Question: What is the characteristic polynomial of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$?

Definition
The **trace** of a square matrix A is $\mathrm{Tr}(A)$ = sum of the diagonal entries of A.

Shortcut
The characteristic polynomial of a 2×2 matrix A is
$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\,\lambda + \det(A)$.

trace A

det A.

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - bc$$

$$= ad - d\lambda - a\lambda + \lambda^2 - bc$$

$$= \lambda^2 - (a+d)\lambda + ad - bc$$

Summary

We did two different things today.

First we talked about the geometry of eigenvalues and eigenvectors:

- Eigenvectors are vectors v such that v and Av are on the same line through the origin.
- You can pick out the eigenvectors geometrically if you have a picture of the associated transformation.

Then we talked about characteristic polynomials:

- We learned to find the eigenvalues of a matrix by computing the roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.
- For a 2×2 matrix A, the characteristic polynomial is just

$$p(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A).$$

Section 5.4

Diagonalization

Diagonalization or why you should care about eigenvalues, anyway!

Exam 3 next week on Friday 7/17 @ 2pm in lecture hall IC 211

Motivation

Difference equations

Many real-word linear algebra problems have the form:

$$v_1 = Av_0, \quad v_2 = Av_1 = A^2v_0, \quad v_3 = Av_2 = A^3v_0, \quad \dots \quad v_n = Av_{n-1} = A^nv_0.$$

This is called a **difference equation**.

Our toy example about rabbit populations had this form.

The question is, what happens to v_n as $n \to \infty$?

- Taking powers of diagonal matrices is easy!
- Taking powers of *diagonalizable* matrices is still easy!
- Diagonalizing a matrix is an eigenvalue problem.

Powers of Diagonal Matrices

If D is diagonal, then D^n is also diagonal; its diagonal entries are the nth powers of the diagonal entries of D:

$$D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \quad D^2 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \quad D^3 = \begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}, \quad \dots \quad D^n = \begin{pmatrix} 2^n & 0 \\ 0 & (-1)^n \end{pmatrix}.$$

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad D^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{9} \end{pmatrix}, \quad D^3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{27} \end{pmatrix}, \quad \dots \quad D^n = \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & \frac{1}{2^n} & 0 \\ 0 & 0 & \frac{1}{3^n} \end{pmatrix}$$

$$D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$D^2 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2^2 & 0 \\ 0 & (-1)^2 \end{bmatrix}$$

$$D^3 = \overbrace{\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}}^{D}\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}}_{D^2} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2^3 & 0 \\ 0 & (-1)^3 \end{bmatrix}$$

$$D^n = \begin{bmatrix} 2^n & 0 \\ 0 & (-1)^n \end{bmatrix}.$$

Similar Matrices

Definition

Two $n \times n$ matrices are **similar** if there exists an invertible $n \times n$ matrix C such that $A = CBC^{-1}$.

A is similar to diagonal D if can find new invertible matrix C

such that

$$\boxed{A = CDC^{-1}}$$

Fact: If two matrices are similar then so are their powers:

$$A = CBC^{-1} \implies A^n = CB^nC^{-1}.$$

$$A^2 = (CDC^{-1})(CDC^{-1})$$
$$= CD\cancel{C^{-1}}\cancel{C}DC^{-1}$$
$$= CDIDC^{-1}$$
$$= CD^2C^{-1}$$

$$A^3 = CD\cancel{C^{-1}} * \cancel{C}D\cancel{C^{-1}} * \cancel{C}DC^{-1}$$
$$= CDDDC^{-1} = CD^3C^{-1}$$

$$\boxed{A^n = CD^nC^{-1}}$$

for n=3 and n=6

Powers of Matrices that are Similar to Diagonal Ones

What if A is not diagonal?

Example

Let $A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}$. Compute A^n, using

$$A = CDC^{-1} \quad \text{for} \quad C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$A^3 = \begin{bmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{bmatrix}^3$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}^3\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}\left(\frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}\right)$$

$$= \frac{-1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\underbrace{\begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}} = \frac{-1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\overbrace{\begin{pmatrix} -8 & -8 \\ 1 & -1 \end{pmatrix}}$$

$$= \frac{-1}{2}\begin{bmatrix} -7 & -9 \\ -9 & 7 \end{bmatrix} = \boxed{\begin{bmatrix} 7/2 & 9/2 \\ 9/2 & -7/2 \end{bmatrix}}$$

(continued on next page)

for n=3 and n=6

Powers of Matrices that are Similar to Diagonal Ones

What if A is not diagonal?

Example

Let $A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}$. Compute A^n, using

$$A = CDC^{-1} \quad \text{for} \quad C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$A^6 = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}$$

$\lambda_1 = 2 \quad \lambda_2 = -1$ (eigenvalues)

$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (eigenvectors) ✓

recall

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^6 = \begin{bmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{bmatrix}^6$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}^6\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 64 & 0 \\ 0 & 1 \end{pmatrix} * -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -64 & -64 \\ -1 & 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -65 & -63 \\ -63 & -65 \end{pmatrix} = \begin{pmatrix} 65/2 & 63/2 \\ 63/2 & 65/2 \end{pmatrix}$$

A^6

Sanity check:

$$Av_1 = \begin{bmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad (\lambda = 2)$$

$$Av_2 = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{bmatrix} 1/2 - 3/2 \\ 3/2 - 1/2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad (\lambda = -1)$$

Diagonalizable Matrices

Definition

An $n \times n$ matrix A is **diagonalizable** if it is similar to a diagonal matrix:

$$A = CDC^{-1} \quad \text{for } D \text{ diagonal.}$$

Important

If $A = CDC^{-1}$ for $D = \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix}$ then

$$A^k = CD^kC^{-1} = C\begin{pmatrix} d_{11}^k & 0 & 0 \\ 0 & d_{22}^k & 0 \\ 0 & 0 & d_{33}^k \end{pmatrix}C^{-1}.$$

$$\left(\begin{matrix} v_i \neq 0 \\ Av_i = \lambda_i v_i \end{matrix} \right)$$

Diagonalization

The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors in $\mathbf{R}^n$.

In this case, $A = CDC^{-1}$ for

$$C = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where $v_1, v_2, \ldots, v_n$ are linearly independent eigenvectors, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the corresponding eigenvalues (in the same order).

FACT: If $\lambda_1, \lambda_2, \lambda_3$ distinct eigenvalues then $\{v_1, v_2, v_3\}$ are linearly ind.

Corollary ← a theorem that follows easily from another theorem

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Diagonalization

Procedure

How to diagonalize a matrix A:

1. Find the eigenvalues of A using the characteristic polynomial.
2. For each eigenvalue λ of A, compute a basis $\mathcal{B}_\lambda$ for the λ-eigenspace.
3. If there are fewer than n total vectors in the union of all of the eigenspace bases $\mathcal{B}_\lambda$, then the matrix is not diagonalizable.
4. Otherwise, the n vectors $v_1, v_2, \ldots, v_n$ in your eigenspace bases are linearly independent, and $A = CDC^{-1}$ for

$$C = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where λ_i is the eigenvalue for v_i.

Warning! Knowing A has **repeated** eigenvalues does **NOT** mean that it is **not** diagonalizable!

If A has repeated eigenvalues A may or may not be diag'ble.

Diagonalization
Example

Problem: Diagonalize $A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}$.

The characteristic polynomial is
$$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\,\lambda + \det(A) = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2).$$

$\lambda_1 = 2 \quad \lambda_2 = -1$

$v_1 = \begin{pmatrix}1\\1\end{pmatrix} \quad v_2 = \begin{bmatrix}1\\-1\end{bmatrix}$

$\frac{1}{4} - \frac{9}{4} = -\frac{8}{4}$
$= -2$

Q: How to diagonalize? $A = CDC^{-1}$, C invertible cols are eigenvectors of A

& D diagonal w/ λ's on diagonal entries.

A: Find eigenvalues & eigenvectors.

Start w/ char poly.

$$p(\lambda) = \det(A - \lambda I) = \lambda^2 - \mathrm{trace}(A)\lambda + \det A$$
$$= \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$$

get $\lambda_1 = 2,\ \lambda_2 = -1.$ eigenvalues.

next get eigenvectors by solving $(A - \lambda I)x = 0$.

$\lambda = 2$

$$A - 2I = \begin{bmatrix} 1/2 - 2 & 3/2 \\ 3/2 & 1/2 - 2 \end{bmatrix} = \begin{bmatrix} -3/2 & 3/2 \\ 3/2 & -3/2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

x_2

$x = x_2\begin{pmatrix}1\\1\end{pmatrix}$ So $v_1 = \begin{pmatrix}1\\1\end{pmatrix}$.

$\lambda = -1$

$$A - (-1)I = A + I = \begin{pmatrix} 3/2 & 3/2 \\ 3/2 & 3/2 \end{pmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

x_2

$x = x_2\begin{pmatrix}-1\\1\end{pmatrix}$ So $v_2 = \begin{pmatrix}-1\\1\end{pmatrix}$

So $A = CDC^{-1}$

$$\boxed{A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}}$$

also ok: $= \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$

just match the order for λ's entries of D & v_i's cols of C.

Diagonalization
Another example

Problem: Diagonalize $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

The characteristic polynomial is
$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -(\lambda-1)^2(\lambda-2).$$

Algebraic Multiplicity

Definition

The **(algebraic) multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

Diagonalization
Another example

Problem: Diagonalize $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

The characteristic polynomial is

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -(\lambda - 1)^2(\lambda - 2).$$

Algebraic Multiplicity

Definition

The **(algebraic) multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

So eigenvalues are $\lambda_1 = 1 \quad \lambda_2 = 1 \quad \lambda_3 = 2$

algebraic mult. of $\lambda_1 = 1$ is $\boxed{2}$

alg. mult. of $\lambda_3 = 2$ is $\boxed{1}$

META: ① Find λ's by solving

$$p(\lambda) = \det(A - \lambda I) = 0$$

to get eigenvalues

② For each eigenvalue row reduce $A - \lambda I$ & get parametric vector form of solns to $(A - \lambda I)x = 0$ to get a basis of eigenvectors.

③ Form the matrices

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \& \quad C = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$$

(& make sure to match order of columns of C w/ entries of D)

$\lambda_1 = 1$ (alg mult 2)

$$A - I = \begin{bmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\begin{array}{cccc} & x_2 & x_3 \\ \sim \left[\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} -1 \\ 0 \\ 0 \end{matrix} & \left.\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right] \end{array}$$

$x_1 = x_2$

$x_2 = x_2$ (free)

$x_3 = x_3$ (free)

$$X = \begin{pmatrix} x_2 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

So $\underbrace{v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \; v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{basis for } \lambda = 1 \text{ eigenspace.}}$

$\lambda_2 = 2$ (alg mult 1).

$$A - 2I = \begin{bmatrix} 2 & -3 & 0 \\ 2 & -3 & 0 \\ 1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/2 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3/2 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{array}{c} \hphantom{0 \quad 0 \quad} x_3 \\ \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$X = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad C = \begin{array}{c} \hphantom{1 \quad 0 \quad} v_3 \downarrow ?? \\ \begin{bmatrix} 1 & 0 & -3 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$A = \begin{pmatrix} 1 & 0 & -3 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{pmatrix} 1 & 0 & -3 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}^{-1}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad C = \begin{pmatrix} 1 & 0 & -3 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$$

$$A = CDC^{-1}$$

Diagonalization
Procedure

The Diagonalization Theorem (Alternate Form)

Let A be an $n \times n$ matrix. The following are equivalent:

1. A is diagonalizable.
2. The sum of the geometric multiplicities of the eigenvalues of A equals n.
3. The sum of the algebraic multiplicities of the eigenvalues of A equals n, and for each eigenvalue, *the geometric multiplicity equals the algebraic multiplicity*.

Algebraic Multiplicity

Definition

The **(algebraic) multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

Non-Distinct Eigenvalues

Definition

Let λ be an eigenvalue of a square matrix A. The **geometric multiplicity** of λ is the dimension of the λ-eigenspace.

Theorem

Let λ be an eigenvalue of a square matrix A. Then

$$1 \leq (\text{the geometric multiplicity of } \lambda) \leq (\text{the algebraic multiplicity of } \lambda).$$

if "=" for each λ Then you can diagonalize A.

Non-Distinct Eigenvalues
Examples

Example

If A has n distinct eigenvalues, then the algebraic multiplicity of each equals 1, hence so does the geometric multiplicity, and therefore A is diagonalizable.

For example, $A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}$ has eigenvalues -1 and 2, so it is diagonalizable.

Example

The matrix $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$ has characteristic polynomial

$$f(\lambda) = -(\lambda-1)^2(\lambda-2).$$

The algebraic multiplicities of 1 and 2 are 2 and 1, respectively. They sum to 3. We showed before that the geometric multiplicity of 1 is 2 (the 1-eigenspace has dimension 2). The eigenvalue 2 automatically has geometric multiplicity 1. Hence the geometric multiplicities add up to 3, so A is diagonalizable.

Exponent of root λ in char poly

dim of eigenspace. (# free vars $A-\lambda I$)

Idea: alg $\neq$ geo for at least one eigenvalue λ.

Diagonalization
A non-diagonalizable matrix

Problem: Show that $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

Step 1: eigenvalues are $\lambda_1 = 1$ $\lambda_2 = 1$

$p(\lambda) = \det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 = 0$

$\lambda = 1$ only one root w/ mult. 2.

alg mult. of $\lambda=1$ is $\boxed{2}$

Step 2: how many linearly ind eigenvectors for A?

$\lambda=1$: $A - 1I = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $x_1 = x_1$ free, $x_2 = 0$

$X = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ basis for $\lambda_1 = 1$ eigenspace is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

geo mult. of $\lambda=1$ is $\boxed{1}$

Since alg $\neq$ geo for $\lambda_1 = 1$ The matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is NOT diag'ble.

You Try It!

Poll

Poll

Which of the following matrices are diagonalizable, and why?

A. $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ B. $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ C. $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ D. $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

A. no, not diag'ble $\lambda_1 = 1$ w/ alg mult. 2, but geo mult. 1. $2 \neq 1$.

B. yes, diag'ble. b/c $\lambda_1 = 1$, $\lambda_2 = 2$ distinct eigenvalues $\xrightarrow{\text{Thm}}$ diag'ble

C. no, not diag'ble $\lambda_1 = 2$ w/ alg mult. 2, but geo mult. 1. $2 \neq 1$.

D. yes, diag'ble b/c already $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ diagonal! b/c $A = D$ $C = I$. $A = IDI^{-1}$

More examples!

Problem: Diagonalize the matrix $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$\lambda_1 = 3, \lambda_2 = 2, \lambda_1 = 1$

alg = geo = 1 for each one.

$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$C = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$

↑ get these by row reducing $A - \lambda I$.

Problem: Diagonalize the matrix $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$ which has characteristic polynomial $p(\lambda) = -\lambda^3 - 3\lambda^2 + 4 = -(\lambda - 1)(\lambda + 2)^2$

$A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$

$\lambda_1 = 1$, $\lambda_2 = -2$ alg 2

alg 1.

↑ geo 2 or not ??

You Try It!

Poll

Poll

Which of the following matrices are diagonalizable, and why?

A. $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ B. $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ C. $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ D. $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Summary

- A matrix A is **diagonalizable** if it is similar to a diagonal matrix D: $A = CDC^{-1}$.
- It is easy to take powers of diagonalizable matrices: $A^r = CD^rC^{-1}$.
- An $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors $v_1, v_2, \ldots, v_n$ in $\mathbf{R}^n$, in which case $A = CDC^{-1}$ for

$$C = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

- If A has n distinct real eigenvalues, then it is diagonalizable.
- The **geometric multiplicity** of an eigenvalue λ is the dimension of the λ-eigenspace.
- $1 \leq$ (geometric multiplicity) $\leq$ (algebraic multiplicity).
- An $n \times n$ matrix is diagonalizable if and only if the sum of the geometric multiplicities is n.