EXAM 2 THIS WEEK
On Friday, July 17 @ 2pm in the lecture room IC 211.

Section 5.5

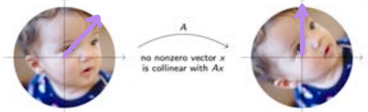
Complex Eigenvalues

A Matrix with No Eigenvectors

Consider the matrix for the linear transformation for rotation by $\pi/4$ in the plane. The matrix is:

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

This matrix has no eigenvectors, as you can see geometrically: [interactive]



Question: what are the eigenvalues of A? maybe there are none?

Hint: compute the characteristic polynomial!

Compute char poly

$p(\lambda) = \det(A - \lambda I)$

$= \det\left(\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)$

$= \lambda^2 - \frac{2}{\sqrt{2}}\lambda + 1$

$= \lambda^2 - \sqrt{2}\lambda + 1 = 0$

Solve using quad formula

$\lambda = \frac{\sqrt{2} \pm \sqrt{2-4}}{2}$

$= \frac{\sqrt{2}}{2} \pm \frac{\sqrt{-2}}{2}$

$= \frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}\cdot\sqrt{-1}}{2}$

$= \boxed{\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2} i}$

two eigenvalues of A

$\lambda_1 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$

$\lambda_2 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$

Operations on Complex Numbers

Definition
The number i is defined such that $i^2 = -1$.

① Addition: $(2 - 3i) + (-1 + i) = (2-1) + (-3+1)i$ (real parts)

$= \boxed{1 - 2i}$

② Multiplication: $(2-3i)(-1+i) = -2 + 2i + 3i - 3i^2$ (FOIL)

$= -2 + 5i - 3i^2$ (real part, imaginary part)

$= -2 + 5i - 3(-1) = \boxed{1 + 5i}$ (simplify $i^2 = -1$)

Complex conjugation: $\overline{a + bi} = a - bi$ is the **complex conjugate** of $a + bi$.

③ Example: $\overline{2 - 3i} = 2 + 3i$

④ $\overline{-1 + i} = -1 - i$

FACT: $(a+bi)(a-bi) = a^2 - i^2 b^2$

$= a^2 + b^2 \geq 0$ (real)

⑤ LHS ✓ RHS

$\overline{(2-3i)(-1+i)} \stackrel{?}{=} \overline{(2-3i)}\,\overline{(-1+i)}$

FACT, $\overline{z_1 * z_2} = \overline{z_1} * \overline{z_2}$

LHS $\overline{(2-3i)(-1+i)} = \overline{1+5i} = 1 - 5i$ ✓

LHS=RHS

RHS $(2+3i)(-1-i) = -2 - 2i - 3i - 3i^2$

$= -2 + 3 - 5i = 1 - 5i$ ✓

⑥ Division using conjugates

Example:

$$\frac{1+i}{1-i} = \frac{1+i}{1-i} * \frac{1+i}{1+i} = \frac{1+2i+i^2}{1-i^2} = \frac{2i}{2} = \boxed{i}$$

* conjugate of bottom on top & bottom

The Fundamental Theorem of Algebra

The whole point of using complex numbers is to solve polynomial equations. It turns out that they are enough to find all solutions of all polynomial equations:

Fundamental Theorem of Algebra

Every polynomial of degree n has exactly n complex roots, counted with multiplicity.

Equivalently, if $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ is a polynomial of degree n, then

$$f(x) = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$$

for (not necessarily distinct) complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Important

If f is a polynomial with *real* coefficients, and if λ is a complex root of f, then so is $\bar{\lambda}$:

$$0 = \overline{f(\lambda)} = \overline{\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0}$$
$$= \bar{\lambda}^n + a_{n-1}\bar{\lambda}^{n-1} + \cdots + a_1\bar{\lambda} + a_0 = f(\bar{\lambda}).$$

Therefore complex roots of real polynomials come in *conjugate pairs*.

Consequently,

If λ is an eigenvalue of a real matrix A then so is $\bar{\lambda}$ the conjugate.

META: ① compute $A-\lambda I$ & row reduce to RREF
② get parametric vector form for soln to $(A-\lambda I)x=0$

Example: find an eigenvector for each eigenvalue.

$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ has eigenvalues $\boxed{\lambda = \frac{1 \pm i}{\sqrt{2}}}$.

$\lambda_1 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ Notice $\bar{\lambda}_1 = \lambda_2$

$\lambda_2 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ ✓

$\lambda_1 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

$$A - \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)I = \underbrace{\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}}_{A} - \underbrace{\begin{bmatrix} \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i & 0 \\ 0 & \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \end{bmatrix}}_{\lambda I}$$

$A - \lambda I$

$\begin{bmatrix} 1 & 3 \\ \cancel{4} & \cancel{12} \end{bmatrix}$

$$= \begin{bmatrix} -\frac{\sqrt{2}}{2}i & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}i \end{bmatrix} \sim \updownarrow \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2\, i \\ -\frac{\sqrt{2}}{2}i & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\sim \begin{array}{l} \frac{2}{\sqrt{2}}R_1 \to \\ \frac{2}{\sqrt{2}}R_2 \to \end{array} \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix} \sim \boxed{i}R_1 + R_2 \to \begin{bmatrix} 1 & -i \\ 0 & -i \cdot i - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -i \\ 0 & \cancel{2}\,0 \end{bmatrix} \;???$$

x_2

$\begin{cases} x_1 - ix_2 = 0 \\ x_2 = x_2 \text{ (free)} \end{cases}$ $\begin{cases} x_1 = ix_2 \\ x_2 = x_2 \text{ (free)} \end{cases}$ $X = \begin{bmatrix} ix_2 \\ x_2 \end{bmatrix} \Rightarrow X = x_2\begin{bmatrix} i \\ 1 \end{bmatrix}$

$\begin{bmatrix} i \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_1 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

$\lambda_2 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

$X = x_2\begin{bmatrix} -i \\ 1 \end{bmatrix}$ $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ is an eigenvector for λ_2

$$A - \lambda I = \begin{bmatrix} \sqrt{2}/2 - \left(\sqrt{2}/2 - \frac{\sqrt{2}}{2}i\right) & -\sqrt{2}/2 \\ \sqrt{2}/2 & +\frac{\sqrt{2}}{2}\cancel{\lambda} - \left(\sqrt{2}/2 - \frac{\sqrt{2}}{2}i\right) \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2\, i & -\sqrt{2}/2 \\ \sqrt{2}/2 & \frac{\sqrt{2}}{2}i \end{bmatrix} \sim \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2\, i \\ 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$$

Fact

Let A be a real square matrix. If λ is a complex eigenvalue with eigenvector v, then $\bar{\lambda}$ is an eigenvalue with eigenvector $\bar{v}$.

Why?

$$Av = \lambda v \implies A\bar{v} = \overline{Av} = \overline{\lambda v} = \bar{\lambda}\bar{v}.$$

Both eigenvalues and eigenvectors of real square matrices occur in conjugate pairs.

Problem: Find the eigenvalues of A and an eigenvector for each eigenvalue where $A = \begin{pmatrix} 1 & -2 \\ 5 & -5 \end{pmatrix}$

META: ① Set $\det(A-\lambda I)=0$ & solve for λ to find eigenvalues

② row reduce $A-\lambda I$ to RREF for λ_1

③ get parametric vector form for solns to $(A-\lambda I)x=0$

④ use tricks $\overline{\lambda_1}=\lambda_2$, $\overline{v_1}=v_2$.

① $\det\begin{pmatrix} 1-\lambda & -2 \\ 5 & -5-\lambda \end{pmatrix} = \lambda^2+4\lambda+5=0$

$\Rightarrow \lambda = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm \frac{\sqrt{-4}}{2}$
$= -2 \pm i$

Choose either one.

② $\lambda_1 = -2-i$

$A-(-2-i)I = \begin{bmatrix} 1-(-2-i) & -2 \\ 5 & -5-(-2-i) \end{bmatrix} = \begin{bmatrix} 3+i & -2 \\ 5 & -3+i \end{bmatrix} \sim \begin{bmatrix} 5 & -3+i \\ 3+i & -2 \end{bmatrix}$

$\sim \frac{1}{5}R_1 \to \begin{bmatrix} 1 & \frac{-3+i}{5} \\ 3+i & -2 \end{bmatrix} \sim_{-(3+i)R_1+R_2} \begin{bmatrix} 1 & \frac{-3+i}{5} \\ 0 & -\frac{3+i}{5}(-3-i)-2 \end{bmatrix}$

$= \begin{bmatrix} 1 & -\frac{3+i}{5} \\ 0 & 0 \end{bmatrix}$

$-\frac{3+i}{5}(-3-i)-2 = \frac{+9-i^2}{5}-2 = \frac{9+1}{5}-2 = \frac{10}{5}-2$
$=2-2=0$

$X = x_2 \begin{bmatrix} (3-i)/5 \\ 1 \end{bmatrix}$ So

$\lambda_1 = -2-i$ $V_1 = \begin{bmatrix} \frac{3}{5}-\frac{1}{5}i \\ 1 \end{bmatrix}$

$\lambda_2 = -2+i$ $V_2 = \begin{bmatrix} \frac{3}{5}+\frac{1}{5}i \\ 1 \end{bmatrix}$

also ok?

You Try It!

Problem: Find the eigenvalues of A and an eigenvector for each eigenvalue where $A = \begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix}$

$p(\lambda) = \lambda^2+1=0$ $\quad \lambda_1 = -i$
$\lambda_2 = i$

$\lambda_1 = -i$

$A-(-i)I = \begin{bmatrix} 3 & -5 \\ 2 & -3 \end{bmatrix} + \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$

$= \begin{bmatrix} 3+i & -5 \\ 2 & -3+i \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{-3+i}{2} \\ 0 & 0 \end{bmatrix}$

$X = x_2 \begin{bmatrix} \frac{3-i}{2} \\ 1 \end{bmatrix}$ $\quad V_1 = \begin{bmatrix} \frac{3-i}{2} \\ 1 \end{bmatrix}$ $\quad W_1 = \begin{bmatrix} 5 \\ 3+i \end{bmatrix}$ ✓

$(3+i)\begin{bmatrix} \frac{3-i}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{9-i^2}{2} \\ 3+i \end{bmatrix} = \begin{pmatrix} 5 \\ 3+i \end{pmatrix}$

$\lambda_2 = i$ $\quad V_2 = \begin{bmatrix} \frac{3+i}{2} \\ 1 \end{bmatrix}$ or $W_2 = \begin{bmatrix} 5 \\ 3-i \end{bmatrix}$ both v_1 or v_2 work.

① True or False? If A is a real 2×2 matrix with eigenvalue $\lambda_1 = 5$, then A cannot have eigenvalue $\boxed{\lambda_2 = 2 + 3i.}$

If $\lambda_2 = 2+3i$ is an eigenvalue of real 2×2 matrix A then $\overline{\lambda_2} = 2-3i$ has to be the other eigenvalue.

② True or False? If A is a real 2×2 matrix with eigenvalue $\lambda_1 = 5 + 3i$, then A cannot have eigenvalue $\lambda_2 = \cancel{2+3i}$ $5-3i$

Only choice.

Summary

- One can do arithmetic with complex numbers just like real numbers: add, subtract, multiply, divide.
- An $n \times n$ matrix always exactly has complex n eigenvalues, counted with (algebraic) multiplicity.
- The complex eigenvalues and eigenvectors of a *real* matrix come in complex conjugate pairs:

$$Av = \lambda v \implies A\bar{v} = \bar{\lambda}\bar{v}.$$

Section 5.6

Stochastic Matrices and PageRank

Stochastic Matrices

Definition
A square matrix A is **stochastic** if all of its entries are nonnegative, and the sum of the entries of each column is 1.

We say A is **positive** if all of its entries are positive.

These arise very commonly in modeling of probabalistic phenomena (Markov chains).

Reminder: Exam 3 on Friday, July 17th @ 2pm in lecture hall.

You'll be responsible for knowing basic facts about stochastic matrices, the Perron–Frobenius theorem, and PageRank, but we will not cover them in depth.

$$A = \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}.$$

30% probability a movie rented from location 3 gets returned to location 2

These are ALL stochastic matrices:

$$\begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Stochastic Matrices
Example

Red Box has kiosks all over where you can rent movies. You can return them to any other kiosk. Let A be the matrix whose ij entry is the probability that a customer renting a movie from location j returns it to location i. For example,

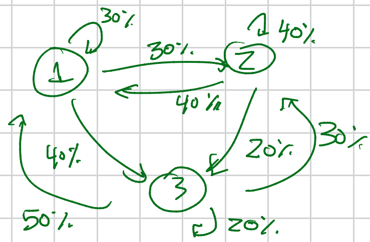
the matrix A says

renting from Box 1 results in
30% returning to Box 1,
30% return to Box 2, and
40% return to Box 3

renting from Box 2 results in
40% return to Box 1,
40% returning to Box 2, and
20% return to Box 3

renting from Box 3 results in
50% return to Box 1,
30% return to Box 2,
20% returning to Box 3.

$$A = \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$$



Try with Ae1, Ae2, Ae3 first!

$$\begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .3 \\ .3 \\ .4 \end{bmatrix} \quad \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .4 \\ .4 \\ .2 \end{bmatrix} \quad \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix}.$$

one movie at box 1.

probabilities to be @ Box1, Box2, or Box 3.

prob moving from Box 2 to other boxes

$$Av = A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} .3x + .4y + .5z \\ .3x + .4y + .3z \\ .4x + .2y + .2z \end{pmatrix}$$

"The number of movies returned to location 2 will be (on average):
30% of the movies from location 1;
40% of the movies from location 2;
30% of the movies from location 3"

#Box 1 = x

#Box 2 = y

#Box 3 = z

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{pmatrix} .3 \\ .3 \\ .4 \end{pmatrix} + y\begin{bmatrix} .4 \\ .4 \\ .2 \end{bmatrix} + z\begin{pmatrix} .5 \\ .3 \\ .2 \end{pmatrix}$$

Stochastic Matrices and Difference Equations

If x_n, y_n, z_n are the numbers of movies in locations 1, 2, 3, respectively, on day n, and $v_n = (x_n, y_n, z_n)$, then:

$$v_n = Av_{n-1} = A^2 v_{n-2} = \cdots = A^n v_0.$$

Recall: This is an example of a **difference equation**.

In general: A difference equation $v_{n+1} = Av_n$ is used to model a state change controlled by a matrix:

- v_n is the "state at time n",
- v_{n+1} is the "state at time $n+1$", and
- $v_{n+1} = Av_n$ means that A is the "change of state matrix."

Matrix Expression and State Vector

$v_0 = \begin{pmatrix} 50 \\ 50 \\ 0 \end{pmatrix}$

$Av_0 = \begin{pmatrix} 35 \\ 35 \\ 30 \end{pmatrix}$

$A^2v_0 = \begin{pmatrix} 39.5 \\ 33.5 \\ 27 \end{pmatrix}$

$A^3v_0 = \begin{pmatrix} 38.75 \\ 33.35 \\ 27.9 \end{pmatrix}$

$A^{10}v_0 \approx \begin{pmatrix} 38.88889 \\ 33.33333 \\ 27.77778 \end{pmatrix}$

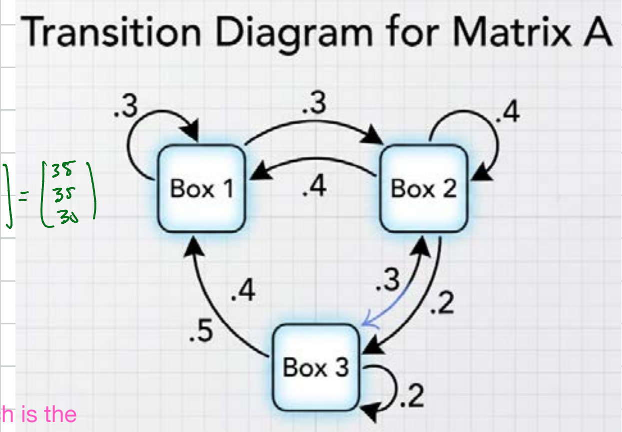
$A^{20}v_0 \approx \begin{pmatrix} 38.88889 \\ 33.33333 \\ 27.77778 \end{pmatrix}$

$$A = \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$$

$V_1 = AV_0$

$$= \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}\begin{bmatrix} 50 \\ 50 \\ 0 \end{bmatrix} = \begin{bmatrix} 15+20 \\ 15+20 \\ 20+10 \end{bmatrix} = \begin{bmatrix} 35 \\ 35 \\ 30 \end{bmatrix}$$

$V_2 = AV_1 = A(AV_0) = A^2 V_0$



Note: the sum of the entries is always N=100 which is the total number of movies in circulation for this problem.

$$\begin{pmatrix} 0.3 & 0.4 & 0.5 \\ 0.3 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.2 \end{pmatrix} \cdot \begin{pmatrix} 50 \\ 50 \\ 0 \end{pmatrix} = \begin{pmatrix} 35 \\ 35 \\ 30 \end{pmatrix}$$

Key observation

Need to find an eigenvector for $\lambda = 1$.

Q: Find the long term trend for the Red Box problem.

Diagonalizable Stochastic Matrices

The Red Box matrix $A = \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$ has characteristic polynomial

$$f(\lambda) = -\lambda^3 + 0.12\lambda - 0.02 = -(\lambda - 1)(\lambda + 0.2)(\lambda - 0.1).$$

$\boxed{\lambda_1 = 1}$ $\quad \lambda_2 = -.2 \quad \lambda_3 = .1 \quad$ (note: $|\lambda_i| \le 1$, $\lambda = 1$ largest one)

META: Step 1 - row reduce A-I and find an eigenvector for lambda=1

$(\lambda = 1)$

RREF of $A - I$

$$A - I = \begin{pmatrix} -0.7 & 0.4 & 0.5 \\ 0.3 & -0.6 & 0.3 \\ 0.4 & 0.2 & -0.8 \end{pmatrix} \sim \begin{pmatrix} -7 & 4 & 5 \\ 3 & -6 & 3 \\ 4 & 2 & -8 \end{pmatrix} \sim \begin{pmatrix} -7 & 4 & 5 \\ 1 & -2 & 1 \\ 4 & 2 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ -7 & 4 & 5 \\ 4 & 2 & -8 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & -10 & 12 \\ 4 & 2 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & -10 & 12 \\ 0 & 10 & -12 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & -10 & 12 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1.2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{bmatrix} 1 & 0 & -1.4 \\ 0 & 1 & -1.2 \\ 0 & 0 & 0 \end{bmatrix}$$

x_3

META: Step 2 - scale your eigenvector so that the sum of the entries equals the total number of objects in the problem (in this case there were N=100 movies in circulation)

$X = x_3 \begin{bmatrix} 1.4 \\ 1.2 \\ 1 \end{bmatrix}$

long term trend is $V = \begin{bmatrix} 1.4 \\ 1.2 \\ 1 \end{bmatrix}$?? have 1.4 movies in Box 1, 1.2 movies in Box 2, 1 movie in Box 3

Step 1: convert V to a prob vector so that entries add to 1.

$1.4 + 1.2 + 1 = 3.6$

$$V = \begin{bmatrix} 1.4/3.6 \\ 1.2/3.6 \\ 1/3.6 \end{bmatrix} = \begin{bmatrix} 14/36 \\ 12/36 \\ 10/36 \end{bmatrix} = \begin{bmatrix} 38.88\% \\ 33.33\% \\ 27.77\% \end{bmatrix}$$

$A^{20}v_0 \approx \begin{pmatrix} 38.88889 \\ 33.33333 \\ 27.77778 \end{pmatrix}$

ANS $\quad 100 \begin{pmatrix} 38.88\% \\ 33.33\% \\ 27.77\% \end{pmatrix} = \boxed{\begin{bmatrix} 38.88 \\ 33.33 \\ 27.77 \end{bmatrix}}$

$[1; 1]$ eigenvector w/ $\lambda=1$

$(1; -1)$ eigenvector w/ $\lambda=1/2$

Diagonalizable Stochastic Matrices

Example from §5.3

Let $A = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$. This is a positive stochastic matrix.

This matrix is diagonalizable:

$$A = CDC^{-1} \quad \text{for} \quad C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Question: Find the long term trend for the initial vector v0=[35 ; 15]

META: Step 1 - row reduce A-I and find an eigenvector for lambda=1

Step 2 - scale your eigenvector so that the sum of the entries equals the total number of objects in the problem (in this case there are N=50 objects in circulation)

x_2

$$A - I = \begin{bmatrix} 3/4-1 & 1/4 \\ 1/4 & 3/4-1 \end{bmatrix} = \begin{bmatrix} -1/4 & 1/4 \\ 1/4 & -1/4 \end{bmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \qquad x = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$V_0 = \begin{bmatrix} 35 \\ 15 \end{bmatrix} \qquad V = \begin{bmatrix} .5 \\ .5 \end{bmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

So long term trend $50\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \boxed{\begin{pmatrix} 25 \\ 25 \end{pmatrix}}$

$$\begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}\begin{bmatrix} 35 \\ 15 \end{bmatrix} = \begin{bmatrix} * \\ * \end{bmatrix}$$

Q: why does this work? Hint: write v0 as a linear combination of eigenvectors and repeatedly apply A.

$$\begin{bmatrix} 35 \\ 15 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad (??)$$

$$\begin{pmatrix} 1 & 1 & | & 35 \\ 1 & -1 & | & 15 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & | & 35 \\ 0 & -2 & | & -20 \end{pmatrix} \sim \begin{bmatrix} 1 & 0 & | & 25 \\ 0 & 1 & | & 10 \end{bmatrix}$$

$$= 25\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 10\begin{bmatrix} 1 \\ -1 \end{bmatrix} \checkmark$$

$[1; 1]$ $\lambda_1 = 1$

$[1; -1]$ $\lambda_2 = 1/2$

$$v_1 = Av_0 = A\begin{bmatrix} 35 \\ 15 \end{bmatrix} = A\left(25\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 10\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$$

$$= 25A\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 10A\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= 25(1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 10\left(\tfrac{1}{2}\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 25+5 \\ 25-5 \end{bmatrix} = \begin{bmatrix} 30 \\ 20 \end{bmatrix}.$$

$$v_n = A^n v_0 = A^n\begin{bmatrix} 35 \\ 15 \end{bmatrix} = A^n\left(25\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 10\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$$

$$= 25A^n\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 10A^n\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= 25(1)^n\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \underbrace{10\left(\tfrac{1}{2}\right)^n\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\text{close to zero}} = \begin{pmatrix} 25+2.5 \\ 25-2.5 \end{pmatrix} = \begin{bmatrix} 27.5 \\ 22.5 \end{bmatrix}$$

$$\boxed{v_n \approx 25\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{pmatrix} 25 \\ 25 \end{pmatrix}}$$

Stochastic Matrices and Difference Equations

If x_n, y_n, z_n are the numbers of movies in locations 1, 2, 3, respectively, on day n, and $v_n = (x_n, y_n, z_n)$, then:

$$v_n = Av_{n-1} = A^2 v_{n-2} = \cdots = A^n v_0.$$

Recall: This is an example of a **difference equation**.

Red Box probably cares about what v_n is as n gets large: it tells them where the movies will end up *eventually*. This seems to involve computing A^n for large n, but as we will see, they actually only have to compute one eigenvector.

In general: A difference equation $v_{n+1} = Av_n$ is used to model a state change controlled by a matrix:

- v_n is the "state at time n",
- v_{n+1} is the "state at time $n+1$", and
- $v_{n+1} = Av_n$ means that A is the "change of state matrix."

Why is it enough to find an eigenvector for λ=1 if you want to find the long term trend?

Eigenvalues of Stochastic Matrices

Fact: 1 is an eigenvalue of a stochastic matrix.

Why? If A is stochastic, then 1 is an eigenvalue of A^T:

$$\begin{pmatrix} .3 & .3 & .4 \\ .4 & .4 & .2 \\ .5 & .3 & .2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Lemma
A and A^T have the same eigenvalues.

Proof: $\det(A - \lambda I) = \det((A - \lambda I)^T) = \det(A^T - \lambda I)$, so they have the same characteristic polynomial.

Eigenvalues of Stochastic Matrices
Continued

Fact: if λ is an eigenvalue of a stochastic matrix, then $|\lambda| \le 1$. Hence 1 is the *largest* eigenvalue (in absolute value).

Diagonalizable Stochastic Matrices
Example from §5.3

Let $A = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$. This is a positive stochastic matrix.

This matrix is diagonalizable:

$$A = CDC^{-1} \quad \text{for} \quad C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Let $w_1 = \binom{1}{1}$ and $w_2 = \binom{1}{-1}$ be the columns of C.

$$A\left(a_1\binom{1}{1} + a_2\binom{1}{-1}\right) = a_1\binom{1}{1} + \frac{1}{2}a_2\binom{1}{-1}$$
$$A^2\left(a_1\binom{1}{1} + a_2\binom{1}{-1}\right) = a_1\binom{1}{1} + \frac{1}{4}a_2\binom{1}{-1}$$
$$A^3\left(a_1\binom{1}{1} + a_2\binom{1}{-1}\right) = a_1\binom{1}{1} + \frac{1}{8}a_2\binom{1}{-1}$$
$$\vdots$$
$$A^n\left(a_1\binom{1}{1} + a_2\binom{1}{-1}\right) = a_1\binom{1}{1} + \frac{1}{2^n}a_2\binom{1}{-1}$$

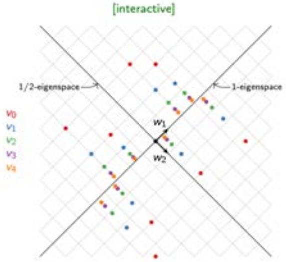
When n is large, the second term disappears, so $A^n x$ approaches $a_1 w_1$, which is an *eigenvector with eigenvalue* 1 (assuming $a_1 \neq 0$). So all vectors get "sucked into the 1-eigenspace," which is spanned by $w_1 = \binom{1}{1}$.

$$A\left(a_1\binom{1}{1} + a_2\binom{1}{-1}\right) = a_1\binom{1}{1} + \frac{1}{2}a_2\binom{1}{-1}$$
$$A^2\left(a_1\binom{1}{1} + a_2\binom{1}{-1}\right) = a_1\binom{1}{1} + \frac{1}{4}a_2\binom{1}{-1}$$
$$A^3\left(a_1\binom{1}{1} + a_2\binom{1}{-1}\right) = a_1\binom{1}{1} + \frac{1}{8}a_2\binom{1}{-1}$$
$$\vdots$$
$$A^n\left(a_1\binom{1}{1} + a_2\binom{1}{-1}\right) = a_1\binom{1}{1} + \frac{1}{2^n}a_2\binom{1}{-1}$$

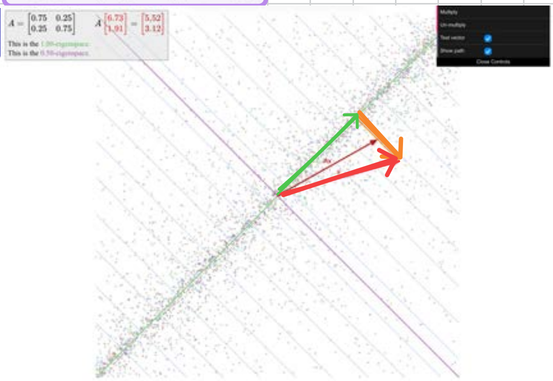
https://textbooks.math.gatech.edu/ila/demos/dynamics.html?mat=1,0:0,1/2&v1=1,1,0&v2=1,-1,0

Diagonalizable Stochastic Matrices
Example, continued

[interactive]



All vectors get "sucked into the 1-eigenspace."



Perron–Frobenius Theorem

Definition
A steady state for a stochastic matrix A is an eigenvector w with eigenvalue 1, such that all entries are positive and sum to 1.

a probability vector w/ eigenvalue $\lambda = 1$.

Perron–Frobenius Theorem
If A is a positive stochastic matrix, then it admits a unique steady state vector w, which spans the 1-eigenspace.

Moreover, for any vector v_0 with entries summing to some number c, the iterates $v_1 = Av_0$, $v_2 = Av_1$, ..., $v_n = Av_{n-1}$, ..., approach cw as n gets large.

Translation: The Perron–Frobenius Theorem says the following:

- The 1-eigenspace of a positive stochastic matrix A is a line.
- To compute the steady state, find any 1-eigenvector (as usual), then divide by the sum of the entries; the resulting vector w has entries that sum to 1, and are *automatically* positive.
- Think of w as a vector of steady state *percentages*: if the movies are distributed according to these percentages today, then they'll be in the same distribution tomorrow.
- The sum c of the entries of v_0 is the total number of movies; eventually, the movies arrange themselves according to the steady state percentage, i.e., $v_n \to cw$.

Let's spend the rest of the day solving sample exam problems!

Spring 2022

d) (2 points) Consider the positive stochastic matrix $A = \begin{pmatrix} 0.2 & 0.6 \\ 0.8 & 0.4 \end{pmatrix}$. The 1-eigenspace of A is spanned by $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$. What vector does $A^n \begin{pmatrix} 17 \\ 4 \end{pmatrix}$ approach as n gets very large?

(i) $\boxed{\begin{pmatrix} 9 \\ 12 \end{pmatrix}}$ (ii) $\begin{pmatrix} 17 \\ 4 \end{pmatrix}$ (iii) $\begin{pmatrix} 51 \\ 16 \end{pmatrix}$ (iv) $\begin{pmatrix} 9/21 \\ 12/21 \end{pmatrix}$ (v) $\begin{pmatrix} 51/67 \\ 16/67 \end{pmatrix}$ (vi) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\lambda = 1$

Idea $A^n \begin{bmatrix} 17 \\ 4 \end{bmatrix} \approx c \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Idea $N = 17 + 4 = 21$

Convert $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ to $\begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$ prob vector

$21 \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \end{bmatrix}$ ans.

$A^n \begin{bmatrix} 17 \\ 4 \end{bmatrix}$ will be close to a $\lambda = 1$ eigenvector but entries need to add up to $N = 17 + 4 = 21$.

b) Amara's coffee shop competes with Ben's cafe for customers. Currently in 2022, Amara has $\boxed{\text{210 customers and Ben has 140 customers.}}$ x

Each year:

A {
- 70% of Amara's customers stay with Amara, while 30% of Amara's customers switch to Ben.
- 60% of Ben's customers stay with Ben, while 40% of Ben's customers switch to Amara.

Answer the following questions.

(i) (4 points) Write a positive stochastic matrix A and a vector x so that Ax is the vector that gives the number of customers for Amara and Ben (in that order) in 2023. Do not carry out the multiplication! Just write A and x.

$A = \begin{bmatrix} .7 & .4 \\ .3 & .6 \end{bmatrix}$ $x = \begin{bmatrix} 210 \\ 140 \end{bmatrix}$

why? $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$ $A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} .4 \\ .6 \end{bmatrix}$

(ii) (3 points) Find the steady-state vector w for the matrix A.

$A - I = \begin{bmatrix} .7-1 & .4 \\ .3 & .6-1 \end{bmatrix} = \begin{bmatrix} -.3 & .4 \\ .3 & -.4 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4/3 \\ 0 & 0 \end{bmatrix}$ (x_2)

$x = x_2 \begin{bmatrix} 4/3 \\ 1 \end{bmatrix}$ (opt 1) $v = \frac{3}{7} \begin{bmatrix} 4/3 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 4/7 \\ 3/7 \end{bmatrix}}$ (opt 2) $x_2 = 3$ $x = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$\frac{4}{3} + 1 = \frac{7}{3}$ $4 + 3 = 7$ $\boxed{v = \begin{bmatrix} 4/7 \\ 3/7 \end{bmatrix}}$

end up at Amara's? 200

end up at Bens? 150

$350 \begin{bmatrix} 4/7 \\ 3/7 \end{bmatrix} = \begin{bmatrix} 200 \\ 150 \end{bmatrix}$ ✓

Spring 2023

c) (3 points) Suppose A is a 2×2 positive stochastic matrix with the property that as n gets very large, $A^n \begin{pmatrix} 90 \\ 70 \end{pmatrix}$ approaches $\boxed{\begin{pmatrix} 120 \\ 40 \end{pmatrix}}$.
What is the steady-state vector w for A?

$\lambda = 1$ eigenvector.

$w = c \begin{pmatrix} 120 \\ 40 \end{pmatrix}$

Problem 7.

Free response. Show your work! A correct answer without sufficient work may receive little or no credit. Parts (a) and (b) are unrelated.

a) Awesome Coffee and Bunk Coffee compete for a market of 60 customers who drink coffee every day. Today, Awesome Coffee has 50 daily customers and Bunk Coffee has 10 daily customers.
Each day:

- 60% of Awesome Coffee customers keep drinking Awesome Coffee, while 40% switch to Bunk Coffee.
- 80% of Bunk Coffee customers keep drinking Bunk Coffee, while 20% switch to Awesome Coffee.

(i) (3 points) Write a positive stochastic matrix A **and** a vector x so that Ax will give the number of customers for Awesome Coffee and Bunk Coffee (in that order) tomorrow. You do not need to compute Ax.

$$A = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \qquad x = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$$

(ii) (4 points) In the long term, approximately how many daily customers will Awesome Coffee have?

To receive full credit, you needed to answer the question that was directly asked. This means that your answer must be a **number**, not a vector. If you just wrote $\begin{pmatrix} 20 \\ 40 \end{pmatrix}$ and did not interpret that vector to answer the question, you did not receive full credit.

Spring 2024

b) (5 points) Axel and Billy are magicians who compete for customers in a group of 200 people. Today, Axel has 160 customers and Billy has 40 customers. Each day:

- 30% of Axel's customers keep attending Axel's show, while 70% of Axel's customers switch to Billy's show.
- 80% of Billy's customers attend Billy's show, while 20% of Billy's customers switch to Axel's show.

(i) Write a positive stochastic matrix B and a vector x so that Bx will give the number of customers for Axel's show and Billy's show (in that order) tomorrow. You do not need to compute Bx.

$$B = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \qquad x = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$$

(ii) Find the steady-state vector w for B. Write your answer in the space below.

$$w = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$$

Spring 2025

4. On this page, you do not need to show your work. Only your answers are graded. Parts (a)-(d) are unrelated.

(a) (3 points) Suppose A is a 6×6 matrix and its characteristic polynomial is

$$\det(A - \lambda I) = \lambda^2(5-\lambda)^3(2-\lambda).$$ $\lambda = 0, 5, 2$ for A.

Which of the following statements are true? Fill in the bubble for all that apply.

○ If the null space of A is a line, then A cannot be diagonalizable.

T/F ● A cannot be a stochastic matrix. $\lambda = 1$ not an eigenvalue

○ The solution set to the equation $Ax = 2x$ is a line.

7. Free response. Show your work except on part (a) part (i)! A correct answer elsewhere without sufficient work will receive little or no credit. Parts (a) and (b) are unrelated.

(a) (7 points) The Great Journey is an online game where participants play either as the Bard or the Cleric. A total of 110 people play the game. This year, 80 players use the Bard and 30 players use the Cleric. Each year:

- 60% of players who use the Bard keep playing as the Bard, while 40% switch to the Cleric.
- 30% of players who use the Cleric keep playing as the Cleric, while 70% switch to the Bard.

(i) Write a positive stochastic matrix A and a vector x so that Ax will give the number of players for the Bard and the Cleric (in that order) next year. You do not need to compute Ax.

$$A = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \qquad x = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$$

(ii) In the long run, roughly how many yearly players will the Cleric have?

Enter your answer here: ____________.

Summary

- **Stochastic** and **positive stochastic** matrices model probabilistic systems.
- We care about the long-term behavior of such a system. This is called the **steady state**. It tells us the eventual state of the system.
- The Perron–Frobenius theorem says that a positive stochastic matrix always has a unique steady state.
- If you can understand the RedBox example, then you understand almost everything.
- The Google matrix is an example of a positive stochastic matrix.
- The steady state of the Google matrix is the PageRank.

Real world application - Google searches!

Google's PageRank

Internet searching in the 90's was a pain. Yahoo or AltaVista would scan pages for your search text, and just list the results with the most occurrences of those words.

Not surprisingly, the more unsavory websites soon learned that by putting the words "Alanis Morissette" a million times in their pages, they could show up first every time an angsty teenager tried to find *Jagged Little Pill* on Napster.

Larry Page and Sergey Brin invented a way to rank pages by *importance*. They founded Google based on their algorithm.

Here's how it works. (roughly)

Reference:

`http://www.math.cornell.edu/~mec/Winter2009/RalucaRemus/Lecture3/lecture3.html`

The Importance Rule

Each webpage has an associated importance, or **rank**. This is a positive number.

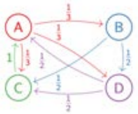
> **The Importance Rule**
>
> If page P links to n other pages $Q_1, Q_2, \ldots, Q_n$, then each Q_i should inherit $\frac{1}{n}$ of P's importance.

- So if a very important page links to your webpage, your webpage is considered important.
- And if a ton of unimportant pages link to your webpage, then it's still important.
- But if only one crappy site links to yours, your page isn't important.

Random surfer interpretation: a "random surfer" just sits at his computer all day, randomly clicking on links. The pages he spends the most time on should be the most important. This turns out to be equivalent to the rank.

The Importance Matrix

Consider the following Internet with only four pages. Links are indicated by arrows.



Page A has 3 links, so it passes $\frac{1}{3}$ of its importance to pages B, C, D.
Page B has 2 links, so it passes $\frac{1}{2}$ of its importance to pages C, D.
Page C has one link, so it passes all of its importance to page A.
Page D has 2 links, so it passes $\frac{1}{2}$ of its importance to pages A, C.

In terms of matrices, if $v = (a, b, c, d)$ is the vector containing the ranks a, b, c, d of the pages A, B, C, D, then

importance matrix: *ij* entry is importance page *j* passes to page *i*

Importance Rule

$$\begin{pmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} c + \frac{1}{2}d \\ \frac{1}{3}a \\ \frac{1}{3}a + \frac{1}{2}b + \frac{1}{2}d \\ \frac{1}{3}a + \frac{1}{2}b \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

Adjustment #1: if any pages have no outward links, then a user picks any page at random (could pick the same page)

Adjustment #2: a user will go to a random page p=15% of the time and click a link 85% of the time.

p=15% is called the **damping factor**

Start with importance matrix A (no Adjustment#1 needed since every page has at least one outward link)

$$A = \begin{pmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{pmatrix}$$

$$G = 0.85 \begin{pmatrix} 0 & 0 & 1 & 0.5 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 0.5 & 0 & 0.5 \\ 1/3 & 0.5 & 0 & 0 \end{pmatrix} + (0.15) \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}$$

$$= 0.85 \begin{pmatrix} 0 & 0 & 1 & 0.5 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 0.5 & 0 & 0.5 \\ 1/3 & 0.5 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0.0375 & 0.0375 & 0.0375 & 0.0375 \\ 0.0375 & 0.0375 & 0.0375 & 0.0375 \\ 0.0375 & 0.0375 & 0.0375 & 0.0375 \\ 0.0375 & 0.0375 & 0.0375 & 0.0375 \end{pmatrix}$$

$$G \approx \begin{pmatrix} 0.0375 & 0.0375 & 0.8875 & 0.4625 \\ 0.32083 & 0.0375 & 0.0375 & 0.0375 \\ 0.32083 & 0.4625 & 0.0375 & 0.4625 \\ 0.32083 & 0.4625 & 0.0375 & 0.0375 \end{pmatrix}$$

Powers of the Google Matrix G^n

Matrix Expression and State Matrix

$$G^1 = \begin{pmatrix} 0.03750 & 0.03750 & 0.88750 & 0.46250 \\ 0.32083 & 0.03750 & 0.03750 & 0.03750 \\ 0.32083 & 0.46250 & 0.03750 & 0.46250 \\ 0.32083 & 0.46250 & 0.03750 & 0.03750 \end{pmatrix}$$

$$G^2 = \begin{pmatrix} 0.44656 & 0.62719 & 0.08531 & 0.44656 \\ 0.04812 & 0.04812 & 0.28896 & 0.16854 \\ 0.32083 & 0.26062 & 0.32083 & 0.20042 \\ 0.18448 & 0.06406 & 0.30490 & 0.18448 \end{pmatrix}$$

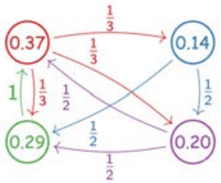
$$G^3 = \begin{pmatrix} 0.38861 & 0.28626 & 0.43979 & 0.28626 \\ 0.16403 & 0.21520 & 0.06167 & 0.16403 \\ 0.26288 & 0.26288 & 0.31406 & 0.31406 \\ 0.18448 & 0.23566 & 0.18448 & 0.23566 \end{pmatrix}$$

$$G^{10} \approx \begin{pmatrix} 0.36815 & 0.36866 & 0.36770 & 0.36844 \\ 0.14166 & 0.14147 & 0.14216 & 0.14182 \\ 0.28808 & 0.28800 & 0.28789 & 0.28783 \\ 0.20211 & 0.20186 & 0.20226 & 0.20192 \end{pmatrix}$$

$$G^{20} \approx \begin{pmatrix} 0.36815 & 0.36815 & 0.36815 & 0.36815 \\ 0.14181 & 0.14181 & 0.14181 & 0.14181 \\ 0.28796 & 0.28796 & 0.28796 & 0.28796 \\ 0.20208 & 0.20208 & 0.20208 & 0.20208 \end{pmatrix}$$

PageRank vector

$$\begin{pmatrix} 0.36815 \\ 0.14181 \\ 0.28796 \\ 0.20208 \end{pmatrix}$$



The Google Matrix

Upshot

Lemma
The Google matrix is a positive stochastic matrix.

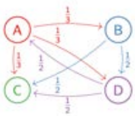
> The PageRank vector is the steady state for the Google Matrix.

This exists and has positive entries by the Perron–Frobenius theorem. The hard part is calculating it: the Google matrix has 1 gazillion rows.

New example: remove the link out of website "C"

The Google Matrix
Example

Consider this Internet:



The importance and modified importance matrices are

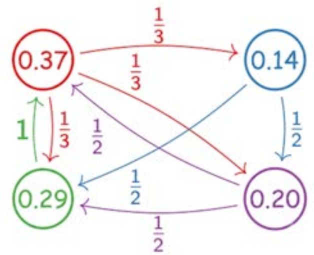
$$A = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{pmatrix} \overset{\text{modify}}{\rightsquigarrow} A' = \begin{pmatrix} 0 & 0 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{4} & 0 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{4} & 0 \end{pmatrix}$$

If we choose the damping factor $p = .15$, then the Google matrix is

$$M = (1-p)A' + pB = \begin{pmatrix} 0.0375 & 0.0375 & 0.2500 & 0.4625 \\ 0.3208 & 0.0375 & 0.2500 & 0.0375 \\ 0.3208 & 0.4625 & 0.2500 & 0.4625 \\ 0.3208 & 0.4625 & 0.2500 & 0.0375 \end{pmatrix}$$

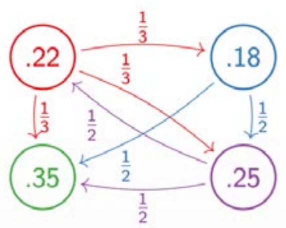
PageRank vector for G w/ link out of "C"

$$\begin{pmatrix} 0.36815 \\ 0.14181 \\ 0.28796 \\ 0.20208 \end{pmatrix}$$



PageRank vector for G with removed link out of "C"

$$\begin{pmatrix} 0.2192 \\ 0.1752 \\ 0.3558 \\ 0.2498 \end{pmatrix}$$

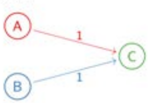


Why do we need a damping factor of p=15%?

Problems with the Importance Matrix
Dangling pages

Observation: the importance matrix is *not* positive: it's only nonnegative. So we can't apply the Perron–Frobenius theorem. Does this cause problems? Yes!

Consider the following Internet:



Adjustment #1: if any pages have no outward links, then a user picks any page at random (could pick the same page)

Replace any columns of zeros with 1/3's

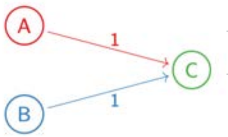
Adjustment #2: a user will go to a random page p=15% of the time and click a link 80% of the time.

Compute G=0.85*A+0.15*M where M has all 1/3's

You Try It!

Problem: Find the importance matrix A for the internet above with three websites A, B, C. Then, find the Google matrix for this internet (it is ok to leave your answer in the form $G = (0.85)A + (0.15)K$. Finally, think about the steps needed to find the steady-state vector for G in order to determine the PageRank vector.

SPOILERS! Answer to previous YouTryIt below.

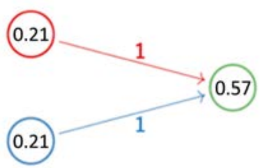


The importance matrix is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$

$$A_* = \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \\ 1 & 1 & \frac{1}{3} \end{pmatrix}$$

$$G = 0.85 \begin{pmatrix} 0 & 0 & 1/3 \\ 0 & 0 & 1/3 \\ 1 & 1 & 1/3 \end{pmatrix} + (0.15) \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

$$G = \begin{pmatrix} 0.05 & 0.05 & 0.3\overline{3} \\ 0.05 & 0.05 & 0.3\overline{3} \\ 0.90 & 0.90 & 0.3\overline{3} \end{pmatrix} \quad q \approx \begin{pmatrix} 0.2128 \\ 0.2128 \\ 0.5745 \end{pmatrix}$$



Why do we need a damping factor of p=15%?

Problems with the Importance Matrix
Disconnected internet

Consider the following Internet:

Adjustment #1: if any pages have no outward links, then a user picks any page at random (could pick the same page)

Replace any columns of zeros with 1/3's

Adjustment #2: a user will go to a random page p=15% of the time and click a link 80% of the time.

Compute G=0.85*A+0.15*M where M has all 1/3's

You Try It!

Problem: Find the importance matrix A for the internet above with five websites A, B, C, D, E. Then, find the Google matrix for this internet (it is ok to leave your answer in the form $G = (0.85)A + (0.15)K$. Finally, think about the steps needed to find the steady-state vector for G in order to determine the PageRank vector.

NOTE: Try to guess the PageRank vector just from the diagram!!!

$$A = A_* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

$$G = 0.85 \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix} + (0.15) \begin{pmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{pmatrix}$$

$$G = \begin{pmatrix} 0.03 & 0.88 & 0.03 & 0.03 & 0.03 \\ 0.88 & 0.03 & 0.03 & 0.03 & 0.03 \\ 0.03 & 0.03 & 0.03 & 0.455 & 0.455 \\ 0.03 & 0.03 & 0.455 & 0.03 & 0.455 \\ 0.03 & 0.03 & 0.455 & 0.455 & 0.03 \end{pmatrix} \quad q = \begin{pmatrix} 0.20 \\ 0.20 \\ 0.20 \\ 0.20 \\ 0.20 \end{pmatrix}$$

Summary

- **Stochastic** and **positive stochastic** matrices model probabilistic systems.
- We care about the long-term behavior of such a system. This is called the **steady state**. It tells us the eventual state of the system.
- The Perron–Frobenius theorem says that a positive stochastic matrix always has a unique steady state.
- If you can understand the RedBox example, then you understand almost everything.
- The Google matrix is an example of a positive stochastic matrix.
- The steady state of the Google matrix is the PageRank.