Section 1.1

Systems of Linear Equations

§1.1

Line, Plane, Space, ...

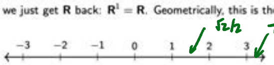
Recall that **R** denotes the collection of all real numbers, i.e. the number line. It contains numbers like $0, -1, \pi, \frac{3}{2}, \ldots$

Definition
Let n be a positive whole number. We define

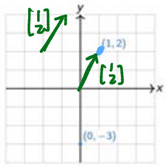
$$\mathbf{R}^n = \text{all ordered } n\text{-tuples of real numbers } (x_1, x_2, x_3, \ldots, x_n).$$

R^n

Example
When $n = 1$, we just get **R** back: $\mathbf{R}^1 = \mathbf{R}$. Geometrically, this is the *number line*.



Example
When $n = 2$, we can think of $\mathbf{R}^2$ as the *plane*. This is because every point on the plane can be represented by an ordered pair of real numbers, namely, its x- and y-coordinates.

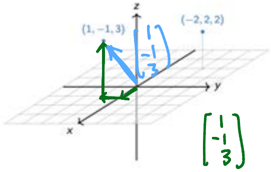


$(1,2)$ location in x-y-plane $\mathbb{R}^2$

$\begin{bmatrix}1\\2\end{bmatrix}$ direction & length or "how to move around in $\mathbb{R}^2$"

"move right 1
& move up 2"

Example
When $n = 3$, we can think of $\mathbf{R}^3$ as the *space* we (appear to) live in. This is because every point in space can be represented by an ordered triple of real numbers, namely, its x-, y-, and z-coordinates.



$\mathbb{R}^3$ Consists of 3-tuples (x,y,z) thought of as locations in 3-space, or $\begin{bmatrix}x\\y\\z\end{bmatrix}$ directions & length

$\begin{bmatrix}1\\-1\\3\end{bmatrix}$ says
First move in x-direction 1 step (forwards)
then move in y-direction -1 step (backwards)
then move in z-direction 3 steps (up)

Line, Plane, Space, . . .
Continued

So what is $\mathbf{R}^4$? or $\mathbf{R}^5$? or $\mathbf{R}^n$?

. . . go back to the *definition:* ordered n-tuples of real numbers

$$(x_1, x_2, x_3, \ldots, x_n).$$

They're still "geometric" spaces, in the sense that our intuition for $\mathbf{R}^2$ and $\mathbf{R}^3$ sometimes extends to $\mathbf{R}^n$, but they're harder to visualize.

We'll make definitions and state theorems that apply to any $\mathbf{R}^n$, but we'll only draw pictures for $\mathbf{R}^2$ and $\mathbf{R}^3$.

The power of using these spaces is the ability to use elements of $\mathbf{R}^n$ to *label* various objects of interest, like solutions to systems of equations.

$\mathbb{R}^2$ all points (x,y) where x,y are real numbers

$\mathbb{R}^3$ all points (x,y,z) where x,y,z are real

You try it!

TRUE or FALSE: The set of points in R^3 consisting of all the points with z-value equal to zero equals the set R^2.

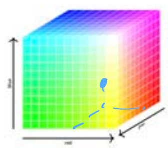
e.g. $(1,1,0)$, $(0,-2,0)$ $(3,-4,0)$ $(\pi, \sqrt{2}, 0)$, . . .

are these in $\mathbb{R}^2$?

all the points have $\underline{\underline{3}}$ components (numbers) so they are all in $\mathbb{R}^3$.

Labeling with $\mathbf{R}^n$
Example

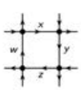
All colors you can see can be described by three quantities: the amount of red, green, and blue light in that color. Therefore, we can use the elements of $\mathbf{R}^3$ to *label* all colors: the point $(.2, .4, .9)$ labels the color with 20% red, 40% green, and 90% blue.



On your phone screen all the pixels are really just a point/vector in $\mathbb{R}^3$

Labeling with $\mathbf{R}^n$
Example

Last time we could have used $\mathbf{R}^4$ to *label* the amount of traffic (x, y, z, w) passing through four streets.

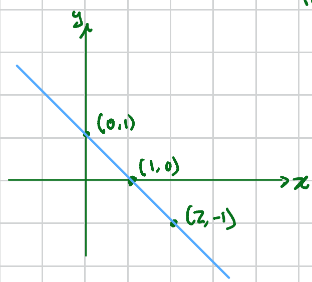


For instance the point $(100, 20, 30, 150)$ corresponds to a situation where 100 cars per hour drive on road x, 20 cars per hour drive on road y, etc.

One Linear Equation

What does the solution set of a linear equation look like?

Example: Graph the implicit equation x+y=1 in R^2 and find an explicit parametrization of the line.

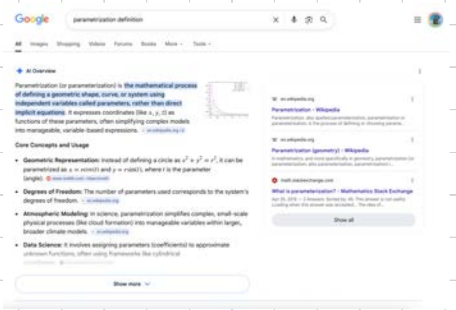


idea: write explicit equation

$\boxed{x+y=1} \Rightarrow y=-x+1$

slope $m=-1$

y-int $b=1$.

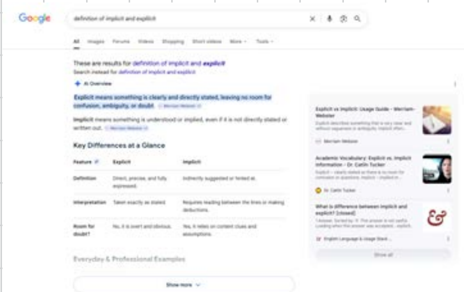


Check: points on line b/c

@ $(0,1)$ $0+1=1$ ✓

@ $(1,0)$ $1+0=1$ ✓

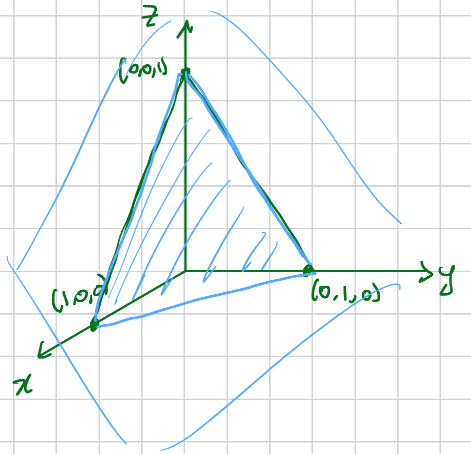
@ $(2,-1)$ $2+(-1)=1$ ✓



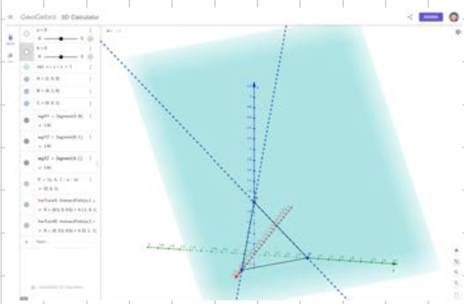
$x+y=1$ equation of the line is implicit.

$y=-x+1$ equation is explicit.

Example: Graph the implicit equation x+y+z=1 in R^3 and find an explicit parametrization of the plane



https://www.geogebra.org/3d/kjrqt66w



e.g. $(2,3,z)$ to be in the plane?

$2+3+z=1 \Rightarrow z=1-5=-4.$

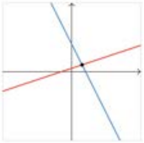
$$\boxed{z=1-x-y, \quad x,y \text{ can be anything}}$$

Systems of Linear Equations

What does the solution set of a *system* of more than one linear equation look like?

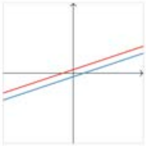
$$x - 3y = -3$$
$$2x + y = 8$$

... is the *intersection* of two lines, which is a *point* in this case.



$$\boxed{\begin{aligned} x - 3y &= -3 \\ x - 3y &= 3 \end{aligned}}$$

has no solution: the lines are *parallel*.



A system of equations with no solutions is called **inconsistent**.

$$\boxed{\begin{aligned} x - 3y &= -3 \\ 2x - 6y &= -6 \end{aligned}}$$

has infinitely many solutions: they are the *same line*.

Note that multiplying an equation by a nonzero number gives the *same solution set*. In other words, they are *equivalent* (systems of) equations.

Natural Questions

* Is there always exactly one solution? No
* how to find the solution(s) systematically
* how to determine IF there are solutions without finding any?

Summary

- $\mathbf{R}^n$ is the set of ordered lists of n numbers. → locations (points) → direction & length (vector)
- $\mathbf{R}^n$ can be used to label geometric objects, like $\mathbf{R}^2$ can label points on a plane.
- The solutions of a system equations look like an intersection of lines, planes, etc.
- Finding all the solutions of a system of equations means finding a **parametric form**: a labeling by some $\mathbf{R}^n$.

$\mathbb{R}^n$ \mathbb{R}^n

R^n

Section 1.2

Row Reduction

Review from 1.1: Solving Systems of Equations

want: a point(s) (x,y,z) which are on ALL 3 planes

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

- A **solution** is a list of numbers $x, y, z, \ldots$ that makes *all* of the equations true.
- The **solution set** is the collection of all solutions.
- **Solving** the system means finding the solution set in a "parameterized" form.

Definition

A system of equations is called **inconsistent** if it has no solution. It is **consistent** otherwise.

Solve:

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

* row reduction
* substitution (solve for one var & sub into other eqns)

→ * elimination (add a multiple of one ~~eqn~~ equation to another)

You try it! What are some techniques that you already know to solve a system like this?

$$\begin{aligned} ①\quad & x+2y+3z=6 \\ ②\quad & 2x-3y+2z=14 \\ ③\quad & 3x+y-z=-2 \end{aligned}$$

$$\begin{aligned} ①\quad & x+2y+3z=6 \\ ①+②\quad & 3x-y+5z=20 \\ ③\quad & 3x+y-z=-2 \end{aligned}$$

$$\begin{aligned} ①\quad & x+2y+3z=6 \\ ②\quad & 3x-y+5z=20 \\ ②+③\quad & 6x+4z=18 \end{aligned}$$

adding two eqns together doesn't change solns. is b/c

and then

Elimination method: in what ways can you manipulate the equations?

doesn't change solns.

- Multiply an equation by a nonzero number. (scale)
- Add a multiple of one equation to another. (replacement)
- Swap two equations. (swap)

doesn't change solns.

doesn't change soln.

Solve:
$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

$\rightsquigarrow$ $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right]$ ↖ manipulate this directly.

You try it! What are some techniques that you already know to solve a system like this?

Elimination method: in what ways can you manipulate the equations?

- Multiply an equation by a nonzero number. (scale)
- Add a multiple of one equation to another. (replacement)
- Swap two equations. (swap)

using matrices	using eqns
$\left[\begin{array}{ccc\|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right]$	$\begin{cases} x + 2y + 3z = 6 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2 \end{cases}$
$\sim \begin{array}{l} -2R_1 + R_2 \to \\ -3R_1 + R_3 \end{array} \left[\begin{array}{ccc\|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right]$	$\Rightarrow \begin{cases} x + 2y + 3z = 6 \\ -7y - 4z = 2 \\ -5y - 10z = -20 \end{cases}$

idea is that the new system of equations will be easier to solve b/c two eqns don't have x's.

$$\begin{aligned} 1x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + 1y - 1z &= -2 \end{aligned} \quad \overset{\text{becomes}}{\rightsquigarrow} \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

← the coefficients of the system of linear equations. & constants on the RHS of equations

This is called an **(augmented) matrix**. Our equation manipulations become **elementary row operations**:

- Multiply all entries in a row by a nonzero number. (scale)
- Add a multiple of each entry of one row to the corresponding entry in another. (row replacement)
- Swap two rows. (swap)

The augmentation bar is a visual aid it's not part of the matrix and whether it's there or not DOESN'T CHANGE any answers about questions you can ask.

Row Equivalence

Definition
Two matrices are called **row equivalent** if one can be obtained from the other by doing some number of elementary row operations.

So the linear equations of row-equivalent matrices have the *same solution set*.

Important
The process of doing row operations to a matrix does not change the solution set of the corresponding linear equations!

$\sim$ \sim

"row equivalence"

You try it!

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right]$$

T/F Is $A = B$? no.

T/F is $A \sim B$? yes. they are row equivalent

$$\sim \begin{array}{l} -2R_1+R_2 \to \\ \\ -3R_1+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right] = B$$

How do you get to a "simple" system?

Row Echelon Form

(REF)

Let's come up with an *algorithm* for turning an arbitrary matrix into a "solved" matrix. What do we mean by "solved"?

> A matrix is in **row echelon form** if
> 1. All zero rows are at the bottom.
> 2. Each leading nonzero entry of a row is to the *right* of the leading entry of the row above.
> 3. Below a leading entry of a row, all entries are *zero*.

leading entries form a "staircase"

They're to the right & below the leading entries above them.

Picture:

leading entries

$$\begin{pmatrix} \boxed{\star} & \star & \star & \star & \star \\ 0 & \boxed{\star} & \star & \star & \star \\ 0 & 0 & 0 & \boxed{\star} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\star$ = any number

$\boxed{\star}$ = any nonzero number

Definition

A **pivot** $\boxed{\star}$ is the first nonzero entry of a row of a matrix in any of it's row echelon forms. A **pivot column** is a column containing a pivot of a matrix *in row echelon form*.

Reduced Row Echelon Form

(RREF)

> A matrix is in **reduced row echelon form** if it is in row echelon form, and in addition,
> 4. The pivot in each nonzero row is equal to 1.
> 5. Each pivot is the only nonzero entry in its column.

Picture:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\star$ = any number

1 = pivot

An Inconsistent Example

Example
Solve the system of equations

$$\begin{aligned} x + y &= 2 \\ 3x + 4y &= 5 \\ 4x + 5y &= 9 \end{aligned}$$

META

① write the system as an augmented matrix $[A|b]$

② row reduce to get the aug. matrix to either REF/RREF

③ rewrite the eqns of the REF/RREF & those simpler equations.

Solve: $\begin{aligned} x + y &= 2 \\ 3x + 4y &= 5 \\ 4x + 5y &= 9 \end{aligned}$

REF/RREF no ✗ ✗

REF/RREF ✓ ✗

① $[A|b] = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 4 & 5 & 9 \end{array}\right] \underset{\substack{-3R_1+R_2 \\ -4R_1+R_3}}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{array}\right] \underset{-R_2+R_3}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array}\right]$ ②

③ $\begin{aligned} x + y &= 2 \\ y &= -1 \\ 0 &= 2 \end{aligned}$ $\left\{\begin{aligned} 1x + 1y &= 2 \\ 0x + 1y &= -1 \\ 0x + 0y &= 2 \end{aligned}\right.$

punchline: once you write down the simple equations you can see that there are NO SOLUTIONS (b/c 3rd equation is NEVER true)

A: So $\boxed{\text{The original system has NO SOLUTIONS and is inconsistent.}}$

1.2: Row Reduction

Theorem
Every matrix is row equivalent to one and only one matrix in reduced row (RREF) echelon form.

Row Echelon Form

Let's come up with an *algorithm* for turning an arbitrary matrix into a "solved" matrix. What do we mean by "solved"?

A matrix is in **row echelon form** if
1. All zero rows are at the bottom.
2. Each leading nonzero entry of a row is to the *right* of the leading entry of the row above.
3. Below a leading entry of a row, all entries are *zero*.

Staircase

Reduced Row Echelon Form

A matrix is in **reduced row echelon form** if it is in row echelon form, and in addition,
4. The pivot in each nonzero row is equal to 1.
5. Each pivot is the only nonzero entry in its column.

Picture:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\star$ = any number
1 = pivot

You try it! REF? RREF?

Poll
Which of the following matrices are in reduced row echelon form?

REF A. $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ B. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ RREF.

none C. $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ D. $\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$ RREF E. $\begin{pmatrix} 0 & 1 & 8 & 0 \end{pmatrix}$ RREF

F. $\left(\begin{array}{cc|c} 1 & 17 & 0 \\ 0 & 0 & 1 \end{array}\right)$ RREF

$x + 17 = 0$
$0 = 1$

Bonus

$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 0 & 1 \end{array}\right]$ REF / ~~RREF~~?

$x + 2y = 2$
$0 = 1$

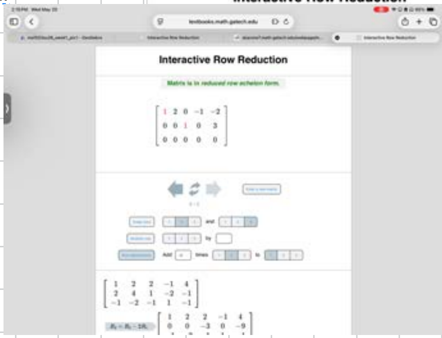
Extra example 1

Translate the equation to an augmented matrix and put the matrix in RREF. Label all pivots. Feel free to use the Interactive Row Reducer.

$$x_1 + 2x_2 + 2x_3 - x_4 = 4$$
$$2x_1 + 4x_2 + x_3 - 2x_4 = -1$$
$$-x_1 - 2x_2 - x_3 + x_4 = -1$$

https://textbooks.math.gatech.edu/ila/demos/rrinter.html

Interactive Row Reduction



Enter this into the interactive row reducer:

1, 2, 2, -1, 4
2, 4, 1, -2, -1
-1, -2, -1, 1, -1

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 4 \\ 2 & 4 & 1 & -2 & -1 \\ -1 & -2 & -1 & 1 & -1 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

RREF

2 pivots, in row 1 - col 1
and row 2 - col 3

Summary

- We can more easily do elimination with matrices. The allowable moves are row swaps, row scales, and row replacements. This is called row reduction.
- A matrix in row echelon form corresponds to a system of linear equations that we can easily solve by back substitution.
- A matrix in reduced row echelon form corresponds to a system of linear equations that we can easily solve just by looking.
- We have an algorithm for row reducing a matrix to reduced row echelon form.
- The reduced row echelon form of a matrix is unique.
- Two matrices that differ by row operations are called row equivalent. Row-equivalent systems have the *same solution set*.
- A system of equations is inconsistent exactly when the corresponding augmented matrix has a pivot in the last column.

(Reduced) Row Echelon Form

1.2 Review

A matrix is in **row echelon form** if

1. All zero rows are at the bottom.
2. Each leading nonzero entry of a row is to the *right* of the leading entry of the row above.
3. Below a leading entry of a row, all entries are *zero*.

A matrix is in **reduced row echelon form** if it is in row echelon form, and in addition,

4. The pivot in each nonzero row is equal to 1.
5. Each pivot is the only nonzero entry in its column.

Row echelon form:

$$\begin{pmatrix} \boxed{\star} & \star & \star & \star & \star \\ 0 & \boxed{\star} & \star & \star & \star \\ 0 & 0 & 0 & \boxed{\star} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Reduced row echelon form:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

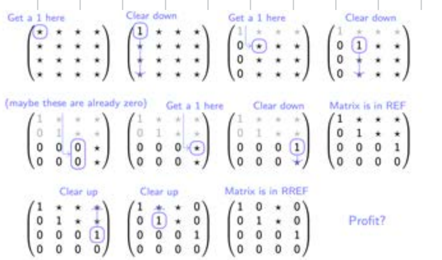
$\boxed{\star}$ = **pivots**

Reinforcing 1.2: Row Reduction Algorithm

Step 1a Swap the 1st row with a lower one so a leftmost nonzero entry is in 1st row (if necessary).
Step 1b Scale 1st row so that its leading entry is equal to 1.
Step 1c Use row replacement so all entries below this 1 are 0.
Step 2a Swap the 2nd row with a lower one so that the leftmost nonzero entry is in 2nd row.
Step 2b Scale 2nd row so that its leading entry is equal to 1.
Step 2c Use row replacement so all entries below this 1 are 0.
Step 3a Swap the 3rd row with a lower one so that the leftmost nonzero entry is in 3rd row.
etc.
Last Step Use row replacement to clear all entries above the pivots, starting with the last pivot (to make life easier).

Example

$$\left(\begin{array}{ccc|c} 0 & -7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{array}\right)$$



Reinforcing 1.2: Row Reduction Algorithm

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Step 1c Use row replacement so all entries below this 1 are 0.

Step 2a Swap the 2nd row with a lower one so that the leftmost nonzero entry is in 2nd row.

Step 2b Scale 2nd row so that its leading entry is equal to 1.

Step 2c Use row replacement so all entries below this 1 are 0.

Step 3a Swap the 3rd row with a lower one so that the leftmost nonzero entry is in 3rd row.

etc.

Last Step Use row replacement to clear all entries above the pivots, starting with the last pivot (to make life easier).

Get a 1 here
$$\begin{pmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$$

Clear down
$$\begin{pmatrix} 1 & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$$

Get a 1 here
$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \end{pmatrix}$$

Clear down
$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \end{pmatrix}$$

(maybe these are already zero)
$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & \star \\ 0 & 0 & 0 & \star \end{pmatrix}$$

Get a 1 here
$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & \star \\ 0 & 0 & 0 & \star \end{pmatrix}$$

Clear down
$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \star \end{pmatrix}$$

Matrix is in REF
$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Clear up
$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Clear up
$$\begin{pmatrix} 1 & \star & \star & 0 \\ 0 & 1 & \star & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Matrix is in RREF
$$\begin{pmatrix} 1 & 0 & \star & 0 \\ 0 & 1 & \star & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Profit?

Example

$$\left(\begin{array}{ccc|c} 0 & -7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

You try it!

Question

What does an augmented matrix in reduced row echelon form look like, if its system of linear equations is inconsistent?

Answer:

$$\left(\begin{array}{cccc|c} 1 & 0 & \star & \star & 0 \\ 0 & 1 & \star & \star & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

pivot in augmented column.

all coefficients zero but constant nonzero.

> An augmented matrix corresponds to an inconsistent system of equations $\boxed{\text{if and only if}}$ *the last* (i.e., the augmented) *column is a pivot column.*

Section 1.3

Parametric Form

Example 1

How do we solve a system of linear equations if the row reduced matrix has a column without a pivot? Let's do an example.

$$\begin{aligned} 2x + y + 12z &= 1 \\ x + 2y + 9z &= -1 \end{aligned} \quad \text{gives rise to the matrix} \quad \left(\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right).$$

https://textbooks.math.gatech.edu/ila/demos/rrinter.html?mat=2,1,12,1:1,2,9,-1

How do you write down the Solutions to a system of linear eqns?

① $[A|b]$

② $[A|b] \sim \cdots$ REF/RREF

③ write eqns. & solve

NOT equality

① $\left[\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ 2 & 1 & 12 & 1 \end{array}\right] \underset{-2R_1+R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ 0 & -3 & -6 & 3 \end{array}\right]$

REF ✓

$\underset{-\frac{1}{3}R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ 0 & 1 & 2 & -1 \end{array}\right] \underset{-2R_2+R_1}{\sim} \begin{array}{ccc} x & y & z\text{(free)} \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right]$ ②

③ $\begin{cases} x + 5z = 1 \\ y + 2z = -1 \end{cases}$

④ Solve for pivot vars in terms of free (non-pivot) vars (don't forget to write free var!!)

$$\boxed{\begin{cases} x = 1 - 5z \\ y = -1 - 2z \\ z = z \text{ (free)} \end{cases}}$$

parametric equation form of the solution.

all solutions.

So for example all the following are solutions

$(z=0)$ $x=1$, $y=-1$, $z=0$

$(z=1)$ $x=-4$, $y=-3$, $z=1$

$(z=7)$ $x=-34$, $y=-15$, $z=7$

Eqn was $2x+y+12z=1$ ✓
$x+2y+9z=-1$

Check @ $(x,y,z)=(1,-1,0)$ ✓

$2-1+0=1$ ✓

$1-2+0=-1$ ✓

Check @ $(x,y,z)=(-4,-3,1)$

$2(-4)+(-3)+12(1) \stackrel{?}{=} 1$ ✓

$(-4)+2(-3)+9(1) = -1$ ✓

Free Variables

Definition

Consider a *consistent* linear system of equations in the variables $x_1, \dots, x_n$. Let A be a row echelon form of the matrix for this system.

We say that x_i is a **free variable** if its corresponding column in A is *not* a pivot column.

Important

1. You can choose *any value* for the free variables in a (consistent) linear system.
2. Free variables come from *columns without pivots* in a matrix in row echelon form.

Example 2

Suppose the reduced row echelon form of the matrix for a linear system in x_1, x_2, x_3, x_4 is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 4 & -1 \end{array}\right)$$

What happened to x_2? What is it allowed to be?

The boxed equation is called the **parametric form** of the general solution to the system of equations. It is obtained by moving all free variables to the right-hand side of the $=$.

① $[A|b]$ ✓

② $[A|b] \sim \cdots$ REF/RREF ✓

③ & ④ write easier system and solve

Note: Two pivot vars x_1, x_3

two free vars x_2, x_4.

$$\begin{cases} 1x_1 + 0x_2 + 0x_3 + 3x_4 = 2 \\ 0x_1 + 0x_2 + 1x_3 + 4x_4 = -1 \end{cases}$$

$$\begin{cases} x_1 + 3x_4 = 2 \\ x_3 + 4x_4 = -1 \end{cases}$$

all solns →

$$\boxed{\begin{aligned} x_1 &= 2 - 3x_4 \\ x_2 &= x_2 \text{ (free)} \\ x_3 &= -1 - 4x_4 \\ x_4 &= x_4 \text{ (free)} \end{aligned}}$$

??

parametric eqn form of solutions (2 free vars)

Check one particular soln

$$\boxed{\begin{aligned} x_1 &= 2 \\ x_2 &= 5 \\ x_3 &= -1 \\ x_4 &= 0 \end{aligned}}$$

You try it??

Example 3

Solve the system of linear equations in x_1, x_2, x_3, x_4:

$$\begin{aligned} x_1 \quad + 5x_3 \quad &= 0 \\ x_4 &= 0 \end{aligned}$$

equality!

x_2 x_3 (free)

①

$$[A|b] = \left[\begin{array}{cccc|c} 1 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 + 5x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

rewrite w/ free vars on RHS & adding free vars as well

parametric eqn form

$$\begin{cases} x_1 = -5x_3 \\ x_2 = x_2 \text{ (free)} \\ x_3 = x_3 \text{ (free)} \\ x_4 = 0 \end{cases}$$

all solns

a particular soln

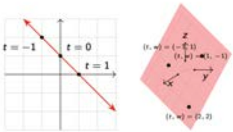
$$\boxed{\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \\ x_4 &= 0 \end{aligned}} \text{ or } \boxed{\begin{aligned} x_1 &= -10 \\ x_2 &= 1 \\ x_3 &= 2 \\ x_4 &= 0 \end{aligned}}$$

Free variables

Geometry, looking ahead!

If we have a consistent system of linear equations, with n variables and k free variables, then the set of solutions is a k-dimensional object in $\mathbb{R}^n$. We will make this precise later, but it is worth thinking about now.

Why does this make sense?



You try it!

Poll

A linear system has 4 variables and 3 equations.

coefficient matrix 3 rows & 4 cols

$$[A|b] = \left[\begin{array}{cccc|c} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{array}\right]$$ augmented matrix 3 rows & 5 cols.

What are the possible solution sets?

- 20% (a) point
- 20% (b) two points
- 80% (c) line
- 70% (d) plane
- 50% (e) 3-dimensional object ✓
- 4 (f) 4-dimensional object ✓

(f) $\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $0=0$, $0=0$, $0=0$ $x_1=$ free, $x_2=$ free, $x_3=$ free, $x_4=$ free 4-dim'l solution set

(e) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $x_1=2$, $x_2=$ free, $x_3=$ free, $x_4=$ free 3-dim'l solution set.

(d) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $x_1=2$, $x_2=3$, $x_3=$ free, $x_4=$ free 2-dim'l solution set

(c) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \end{array}\right]$ $x_1=2$, $x_2=3$, $x_3=4$, $x_4=$ free

(a) not possible b/c # eqns > # vars so couldn't have only pivot cols (at most 1 pivot per row).

(b) NEVER POSSIBLE. only possible # of sol'ns none/exactly one/infinitely many

Yet Another Example

The linear system

$x+y+z=1$ has matrix form $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \end{array}\right)$.

Q: What is the parametric form of the general solution to this linear system of one equation in three variables?

Trichotomy

There are *three possibilities* for the reduced row echelon form of the augmented matrix of a linear system.

1. The last column is a pivot column.
   In this case, the system is *inconsistent*. There are *zero* solutions, i.e. the solution set is *empty*. Picture:
$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$
2. Every column except the last column is a pivot column.
   In this case, the system has a *unique solution*. Picture:
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \star \\ 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \end{array}\right)$$
3. The last column is not a pivot column, and some other column isn't either.
   In this case, the system has *infinitely many* solutions, corresponding to the infinitely many possible values of the free variable(s). Picture:
$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array}\right)$$

Summary

- **Row reduction** is an algorithm for solving a system of linear equations represented by an augmented matrix.
- The goal of row reduction is to put a matrix into **(reduced) row echelon form**, which is the "solved" version of the matrix.
- An augmented matrix corresponds to an inconsistent system if and only if there is a pivot in the augmented column.
- Columns without pivots in the RREF of a matrix correspond to **free variables**. You can assign any value you want to the free variables.
- We can tell whether a linear system has zero, one, or infinitely many solutions using the RREF of the augmented matrix.

So far.

* Systems of linear eqns.
* Matrices & row reducing
* representing solns using parametric eqn. form.