Section 2.3

Matrix Equations

Definition
The **product** of A with a vector x in $\mathbf{R}^n$ is the linear combination

$$Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \overset{\text{def}}{=} x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

this means the equality is a *definition*

these must be equal

The output is a vector in $\mathbf{R}^m$.

Example

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} =$$

Note that the number of columns of A has to equal the number of rows of x.

Matrix Equations
An example

Question
Let v_1, v_2, v_3 be vectors in $\mathbf{R}^3$. How can you write the vector equation

$$2v_1 + 3v_2 - 4v_3 = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}$$

in terms of matrix multiplication?

Matrix Equations
In general

Let $v_1, v_2, \ldots, v_n$, and b be vectors in $\mathbf{R}^m$. Consider the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b.$$

It is equivalent to the **matrix equation**

$$Ax = b$$

where

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Conversely, if A is any $m \times n$ matrix, then

$Ax = b$ is equivalent to the vector equation $x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b$

where $v_1, \ldots, v_n$ are the columns of A, and $x_1, \ldots, x_n$ are the entries of x.

Linear Systems, Vector Equations, Matrix Equations, ...

We now have *four* equivalent ways of writing (and thinking about) linear systems:

1. As a system of equations:
$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 - x_2 &= 5 \end{aligned}$$
2. As an augmented matrix:
$$\left(\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 5 \end{array}\right)$$
3. As a vector equation ($x_1v_1 + \cdots + x_nv_n = b$):
$$x_1\begin{pmatrix}2\\1\end{pmatrix} + x_2\begin{pmatrix}3\\-1\end{pmatrix} = \begin{pmatrix}7\\5\end{pmatrix}$$
4. As a matrix equation ($Ax = b$):
$$\begin{pmatrix}2 & 3\\1 & -1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}7\\5\end{pmatrix}$$

We will move back and forth freely between these over and over again, for the rest of the semester. Get comfortable with them now!

In particular, *all four have the same solution set.*

Very Important Fact That Will Appear on Every Midterm and the Final

$Ax = b$ has a solution

$\iff$ there exist $x_1, \ldots, x_n$ such that $A\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix} = b$

$\iff$ there exist $x_1, \ldots, x_n$ such that $x_1v_1 + \cdots + x_nv_n = b$

$\iff$ b is a linear combination of $v_1, \ldots, v_n$

$\iff$ b is in the span of the columns of A.

"if and only if"

Question

Let $A = \begin{pmatrix}2 & 1\\-1 & 0\\1 & -1\end{pmatrix}$. Does the equation $Ax = \begin{pmatrix}0\\2\\2\end{pmatrix}$ have a solution?

https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=2,-1,1&v2=1,0,-1&target=0,2,2&tlabel=b&range=5

Question

Let $A = \begin{pmatrix}2 & 1\\-1 & 0\\1 & -1\end{pmatrix}$. Does the equation $Ax = \begin{pmatrix}1\\-1\\2\end{pmatrix}$ have a solution?

When Solutions Always Exist

Here are criteria for a linear system to *always* have a solution.

Theorem

Let A be an $m \times n$ (non-augmented) matrix. The following are equivalent:

1. $Ax = b$ has a solution *for all* b in $\mathbf{R}^m$.
2. The span of the columns of A is all of $\mathbf{R}^m$.
3. A has a pivot in each row.

recall that this means that for given A, either they're all true, or they're all false

Example where conditions ARE satisfied

https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=2,-1,1&v2=1,0,-1&v3=-1,2,2&range=5&capopt=matrix

Example where conditions are NOT satisfied

https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=2,-1,1&v2=1,0,-1&v3=1,-1,2&range=5&capopt=matrix

Properties of the Matrix–Vector Product

Let c be a scalar, u, v be vectors, and A a matrix.

- $A(u + v) = Au + Av$
- $A(cv) = cAv$

Consequence: If u and v are solutions to $Ax = 0$, then so is every vector in $\text{Span}\{u, v\}$. Why?

Important

The set of solutions to $Ax = 0$ is a span.

Summary

- We have four equivalent ways of writing a system of linear equations:
  1. As a system of equations.
  2. As an augmented matrix.
  3. As a vector equation.
  4. As a matrix equation $Ax = b$.
- $Ax = b$ is consistent if and only if b is in the span of the columns of A. The latter condition is geometric: you can draw pictures of it.
- $Ax = b$ is consistent for all b in $\mathbf{R}^m$ if and only if the columns of A span $\mathbf{R}^m$.

Section 2.4

Solution Sets

Plan For Today

Today we will learn to describe and draw the solution set of an arbitrary system of linear equations $Ax = b$, using spans.

Recall: the **solution set** is the collection of all vectors x such that $Ax = b$ is true.

Homogeneous Systems

Definition
A system of linear equations of the form $Ax = 0$ is called **homogeneous.**

Definition
A system of linear equations of the form $Ax = b$ with $b \neq 0$ is called **inhomogeneous.**

A homogeneous system always has the solution $x = 0$. This is called the **trivial solution.** The nonzero solutions are called **nontrivial.**

Observation
$Ax = 0$ has a nontrivial solution
$\iff$ there is a free variable
$\iff$ A has a column with no pivot.

Question
What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}?$$

Observation
Since the last column (everything to the right of the $=$) was zero to begin, it will always stay zero! So it's not really necessary to write augmented matrices in the homogeneous case.

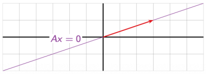
Homogeneous Systems

Example

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}?$$



Note: *one* free variable means the solution set is a *line* in $\mathbf{R}^2$ ($2 = \#$ variables $= \#$ columns).

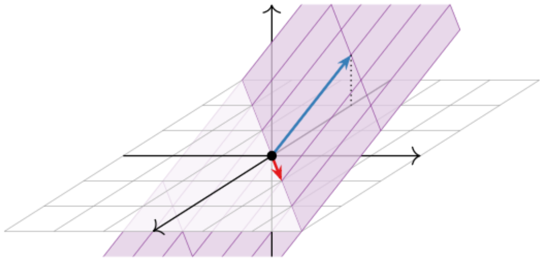
https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?lock=true&x=3,1&mat=1,-3:2,-6&range2=5

Homogeneous Systems
Example

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}?$$



https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?lock=true&x=0,0,0

Note: *two* free variables means the solution set is a *plane* in $\mathbf{R}^3$ ($3 = \#$ variables $= \#$ columns).

Question

What is the solution set of $Ax = 0$, where $A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}$

Note: *two* free variables means the solution set is a *plane* in $\mathbf{R}^4$ ($4 = \#$ variables $= \#$ columns).

Parametric Vector Form

Homogeneous systems

Let A be an $m \times n$ matrix. Suppose that the free variables in the homogeneous equation $Ax = 0$ are, for example, x_3, x_6, and x_8.

1. Find the reduced row echelon form of A.
2. Write the parametric form of the solution set, including the redundant equations $x_3 = x_3$, $x_6 = x_6$, and $x_8 = x_8$. Put equations for all of the x_i in order.
3. Make a single vector equation from these equations by putting x_3, x_6, and x_8 as coefficients of vectors v_3, v_6, and v_8, respectively.

The solutions to $Ax = 0$ will then be expressed in the form

$$x = x_3 v_3 + x_6 v_6 + x_8 v_8$$

for some vectors v_3, v_6, v_8 in $\mathbf{R}^n$, and any scalars x_3, x_6, x_8.

In this case, the solution set to $Ax = 0$ is

$$\text{Span}\{v_3, v_6, v_8\}.$$

Inhomogeneous Systems

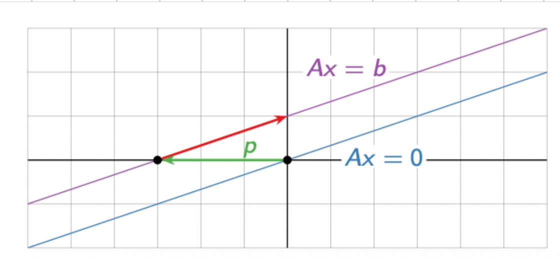
Example

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -3 \\ -6 \end{pmatrix}?$$

https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?x=-3,0&mat=1,-3:2,-6&lock=true&closed=true



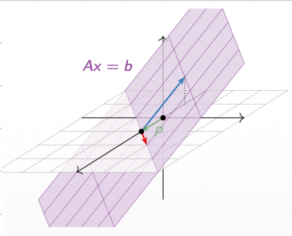
Inhomogeneous Systems

Example

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}?$$



Homogeneous vs. Inhomogeneous Systems

Key Observation

The set of solutions to $Ax = b$, if it is nonempty, is obtained by taking one **specific** or **particular solution** p to $Ax = b$, and adding all solutions to $Ax = 0$.

Very Important

Let A be an $m \times n$ matrix. There are now two *completely different* things you know how to describe using spans:

- The **solution set:** for fixed b, this is all x such that $Ax = b$.
  - This is a span if $b = 0$, or a translate of a span in general (if it's consistent).
  - Lives in $\mathbf{R}^n$.
  - Computed by finding the parametric vector form.
- The **span of the columns:** this is all b such that $Ax = b$ is consistent.
  - This is the span of the columns of A.
  - Lives in $\mathbf{R}^m$.

Summary

- The solution set to a **homogeneous** system $Ax = 0$ is a span. It always contains the **trivial solution** $x = 0$.
- The solution set to a **nonhomogeneous** system $Ax = b$ is either empty, or it is a translate of a span: namely, it is a translate of the solution set of $Ax = 0$.
- The solution set to $Ax = b$ can be expressed as a translate of a span by computing the **parametric vector form** of the solution.
- The solution set to $Ax = b$ and the span of the columns of A (from the previous lecture) are two completely different things, and you have to understand them separately.

Section 2.5

Linear Independence

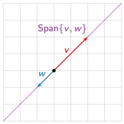
Example: what is

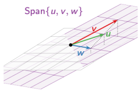
(1) Span{[1;1],[2;2],[0;0]}

(2) Span{{[1;0;1],[0;1;0],[1;1;1]}

Motivation

Sometimes the span of a set of vectors is "smaller" than you expect from the number of vectors.





Linear Independence

Definition

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ in $\mathbf{R}^n$ is **linearly independent** if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution $x_1 = x_2 = \cdots = x_p = 0$. The set $\{v_1, v_2, \ldots, v_p\}$ is **linearly dependent** otherwise.

Like span, linear (in)dependence is another one of those big vocabulary words that you absolutely need to learn. Much of the rest of the course will be built on these concepts, and you need to know exactly what they mean in order to be able to answer questions on quizzes and exams (and solve real-world problems later on).

In other words, $\{v_1, v_2, \ldots, v_p\}$ is linearly dependent if there exist numbers $x_1, x_2, \ldots, x_p$, not all equal to zero, such that

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0.$$

This is called a **linear dependence relation** or an **equation of linear dependence**.

Note that linear (in)dependence is a notion that applies to a *collection of vectors*, not to a single vector, or to one vector in the presence of some others.

Warning: It is not meaningful to say "v is dependent on w", but you can say that the set {v,w} is linearly dependent.

Checking Linear Independence

Question: Is $\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\2\end{pmatrix}, \begin{pmatrix}3\\1\\4\end{pmatrix} \right\}$ linearly independent?

Equivalently, does the (homogeneous) the vector equation

$$x\begin{pmatrix}1\\1\\1\end{pmatrix} + y\begin{pmatrix}1\\-1\\2\end{pmatrix} + z\begin{pmatrix}3\\1\\4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Checking Linear Independence

Question: Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Equivalently, does the (homogeneous) the vector equation

$$x \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + y \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + z \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Linear Independence and Matrix Columns

In general, $\{v_1, v_2, \ldots, v_p\}$ is linearly independent if and only if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution, if and only if the matrix equation

$$Ax = 0$$

has only the trivial solution, where A is the matrix with columns $v_1, v_2, \ldots, v_p$:

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

This is true if and only if the matrix A has a pivot in each column.

Important

- The vectors $v_1, v_2, \ldots, v_p$ are linearly independent if and only if the matrix with columns $v_1, v_2, \ldots, v_p$ has a pivot in each column.
- Solving the matrix equation $Ax = 0$ will either verify that the columns $v_1, v_2, \ldots, v_p$ of A are linearly independent, or will produce a linear dependence relation.

Example: Are the vectors v1=[1;1;0], v2=[2;0;4], and v3=[1;2;-2] linearly dependent? Can you write v3 as a vector in Span{v1,v2}?

Linear Independence

Criterion

Suppose that one of the vectors $\{v_1, v_2, \ldots, v_p\}$ is a linear combination of the other ones (that is, it is in the span of the other ones):

$$v_3 = 2v_1 - \frac{1}{2}v_2$$

Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly *dependent* if and only if one of the vectors is in the span of the other ones.

Equivalently:

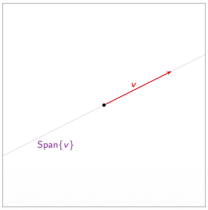
Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly *dependent* if and only if you can remove one of the vectors without shrinking the span.

Translation

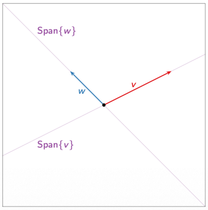
A set of vectors is linearly independent if and only if, every time you add another vector to the set, the span gets bigger.

Linear Independence
Pictures in $\mathbf{R}^2$



One vector $\{v\}$:
Linearly independent if $v \neq 0$.

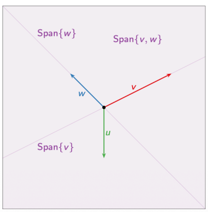
Linear Independence
Pictures in $\mathbf{R}^2$



One vector $\{v\}$:
Linearly independent if $v \neq 0$.

Two vectors $\{v, w\}$:
Linearly independent
- Neither is in the span of the other.
- Span got bigger.

Linear Independence
Pictures in $\mathbf{R}^2$



[interactive 2D: 2 vectors]
[interactive 2D: 3 vectors]

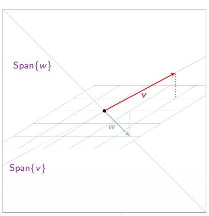
One vector $\{v\}$:
Linearly independent if $v \neq 0$.

Two vectors $\{v, w\}$:
Linearly independent
- Neither is in the span of the other.
- Span got bigger.

Three vectors $\{v, w, u\}$:
Linearly dependent:
- u is in Span$\{v, w\}$.
- Span didn't get bigger after adding u.
- Can remove u without shrinking the span.

Also v is in Span$\{u, w\}$ and w is in Span$\{u, v\}$.

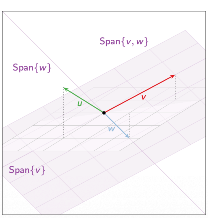
Linear Independence
Pictures in $\mathbf{R}^3$



[interactive 3D: 2 vectors]
[interactive 3D: 3 vectors]

Two vectors $\{v, w\}$:
Linearly independent:
- Neither is in the span of the other.
- Span got bigger when we added w.

Linear Independence
Pictures in $\mathbf{R}^3$



[interactive 3D: 2 vectors]
[interactive 3D: 3 vectors]

Three vectors $\{v, w, u\}$:
Linearly independent: span got bigger when we added u.

https://textbooks.math.gatech.edu/ila/demos/spans.html?captions=indep&v1=2,-1,1&v2=1,0,-1&v3=.5,-.5,1&labels=v,w,x&range=5

Linear Dependence and Free Variables

Theorem

Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbf{R}^n$, and consider the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

Then you can delete the columns of A *without* pivots (the columns corresponding to free variables), without changing $\text{Span}\{v_1, v_2, \ldots, v_p\}$. The pivot columns are linearly independent, so you can't delete any more columns.

This means that each time you add a pivot column, then the span increases.

Example: Which of the vectors in the set {v1,v2,v3,v4,v5} can be removed without changing the span of the vectors?

v1=[1;0;0], v2=[1;1;0], v3=[2;2;0], v4=[1;3;0], v5=[2;2;3]

Linear Independence
Two more facts

Fact 1: Say $v_1, v_2, \dots, v_n$ are in $\mathbf{R}^m$. If $n > m$ then $\{v_1, v_2, \dots, v_n\}$ is linearly *dependent*: the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}.$$

cannot have a pivot in each column (it is too wide).

This says you can't have 4 linearly independent vectors in $\mathbf{R}^3$, for instance.

A wide matrix can't have linearly independent columns.

Example: Is {v1,v2,v3} linearly independent or linearly dependent? If the set is linearly dependent, then find a dependence relation on v1,v2,v3.

v1=[1;1], v2=[0;1], v3=[2;3]

Fact 2: If one of $v_1, v_2, \dots, v_n$ is zero, then $\{v_1, v_2, \dots, v_n\}$ is linearly *dependent*. For instance, if $v_1 = 0$, then

$$1 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 + \cdots + 0 \cdot v_n = 0$$

is a linear dependence relation.

A set containing the zero vector is linearly dependent.

Example: Is {v1,v2,v3} linearly independent or linearly dependent? If the set is linearly dependent, then find a dependence relation on v1,v2,v3.

v1=[1;1;0], v2=[0;0;0], v3=[0;1;1]

Summary

- A set of vectors is **linearly independent** if removing one of the vectors shrinks the span; otherwise it's **linearly dependent**.
- There are several other criteria for linear (in)dependence which lead to pretty pictures.
- The columns of a matrix are linearly independent if and only if the RREF of the matrix has a pivot in every *column*.
- The pivot columns of a matrix A are linearly independent, and you can delete the non-pivot columns (the "free" columns) without changing the span of the columns.
- Wide matrices cannot have linearly independent columns.

Warning

These are not the official definitions!

Linear Independence

Definition

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ in $\mathbf{R}^n$ is **linearly independent** if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution $x_1 = x_2 = \cdots = x_p = 0$. The set $\{v_1, v_2, \ldots, v_p\}$ is **linearly dependent** otherwise.

This is the official textbook definition.