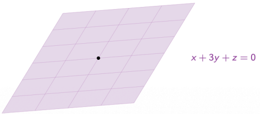
Section 2.6

Subspaces

Motivation

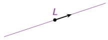
Today we will discuss **subspaces** of $\mathbf{R}^n$.



Examples

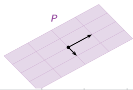
Example

A line L through the origin is a subspace: L contains zero and is easily seen to be closed under addition and scalar multiplication.



Example

A plane P through the origin is a subspace: P contains zero; the sum of two vectors in P is also in P; and any scalar multiple of a vector in P is also in P.



Example

All of $\mathbf{R}^n$: this contains 0, and is closed under addition and scalar multiplication.

Example

The subset $\{0\}$: this subspace contains only one vector.

Definition of Subspace

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V. "not empty"
2. If u and v are in V, then $u + v$ is also in V. "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V. "closed under $\times$ scalars"

A subspace is a span of some set of vectors in it.

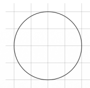
Subsets and Subspaces

They aren't the same thing

A **subset** of $\mathbf{R}^n$ is any collection of vectors in $\mathbf{R}^n$ whatsoever. For example, the unit circle

$$C = \{(x, y) \text{ in } \mathbf{R}^2 \mid x^2 + y^2 = 1\}$$

is a subset of $\mathbf{R}^2$, but it is not a subspace.

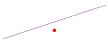


Subset: *yes*
Subspace: *no*

Non-Examples

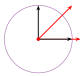
Non-Example

A line L (or any other set) that doesn't contain the origin is not a subspace. Fails: 1.



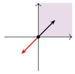
Non-Example

A circle C is not a subspace. Fails: 1,2,3. Think: a circle isn't a "linear space."



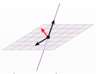
Non-Example

The first quadrant in $\mathbf{R}^2$ is not a subspace. Fails: 3 only.



Non-Example

A line union a plane in $\mathbf{R}^3$ is not a subspace. Fails: 2 only.



Subspaces are Spans, and Spans are Subspaces

Theorem

Any $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is a subspace.

Definition

If $V = \text{Span}\{v_1, v_2, \ldots, v_p\}$, we say that V is the subspace **generated by** or **spanned by** the vectors $v_1, v_2, \ldots, v_p$. We call $\{v_1, v_2, \ldots, v_p\}$ a **spanning set** for V.

!!!

Every subspace is a span, and every span is a subspace.

Example: check from the definitions that Span{v1,v2} is a subspace where v1=[1;0;0] and v2=[0;1;0].

Poll

Poll

Which of the following are subspaces?

A. The empty set $\{\}$.

B. The solution set to a homogeneous system of linear equations.

C. The solution set to an inhomogeneous system of linear equations.

D. The set of all vectors in $\mathbf{R}^n$ with rational (fraction) coordinates.

For the ones which are not subspaces, which property(ies) do they not satisfy?

Poll

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Subspaces

Verification

Let $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbf{R}^2 \mid ab = 0 \right\}$. Let's check if V is a subspace or not.

Definition of Subspace

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

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2. If u and v are in V, then $u + v$ is also in V. "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V. "closed under $\times$ scalars"

Column Space and Null Space

An $m \times n$ matrix A naturally gives rise to *two* subspaces.

Definition

- The **column space** of A is the subspace of $\mathbf{R}^m$ spanned by the columns of A. It is written $\operatorname{Col} A$.
- The **null space** of A is the set of all solutions of the homogeneous equation $Ax = 0$:

$$\operatorname{Nul} A = \{x \text{ in } \mathbf{R}^n \mid Ax = 0\}.$$

This is a subspace of $\mathbf{R}^n$.

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Example: Why is Col(A) always a subspace?

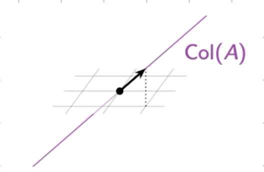
Example: Show that Nul(A) always satisfies all the conditions for being a subspace, using the definitions.

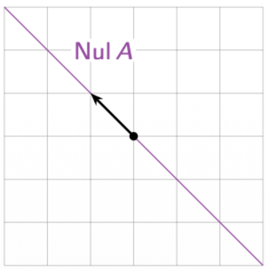
Column Space and Null Space

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Example: Find Col(A) and Nul(A) where A=[1 1 ; 1 1 ; 1 1]





Example: with A as above, describe Col(A) and Nul(A) geometrically.

The Null Space is a Span

The column space of a matrix A is defined to be a span (of the columns).

The null space is defined to be the solution set to $Ax = 0$. It is a subspace, so it is a span.

Question

How to find vectors that span the null space?

Example: Write the null space of the matrix A as a span.
A=[1 0 -1 2 ; 0 1 -2 -3]

Answer: Parametric vector form! We know that the solution set to $Ax = 0$ has a parametric form that looks like

$$x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

if, say, x_3 and x_4 are the free variables. So

$$\text{Nul}\, A = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Subspaces

Summary

- A **subspace** is the same as a span of some number of vectors, but we haven't computed the vectors yet.
- To any matrix is associated two subspaces, the **column space** and the **null space**:

$$\operatorname{Col} A = \text{the span of the columns of } A$$
$$\operatorname{Nul} A = \text{the solution set of } Ax = 0.$$

How do you check if a subset is a subspace?

- Is it a span? Can it be written as a span?
- Can it be written as the column space of a matrix?
- Can it be written as the null space of a matrix?
- Is it all of $\mathbf{R}^n$ or the zero subspace $\{0\}$?
- Can it be written as a type of subspace that we'll learn about later (eigenspaces, ...)?

If so, then it's automatically a subspace.

If all else fails:

- Can you verify directly that it satisfies the three defining properties?

Sections 2.7 and 2.9

Basis, Dimension, Rank and Basis Theorems

Subspaces

Reminder

Recall: a subspace of $\mathbf{R}^n$ is the same thing as a span, except we haven't computed a spanning set yet.

For example, Col A and Nul A for a matrix A.

There are lots of choices of spanning set for a given subspace.

Are some better than others?

Basis of a Subspace

What is the *smallest number* of vectors that are needed to span a subspace?

Definition

Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in V such that:

1. $V = \text{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

The number of vectors in a basis is the **dimension** of V, and is written dim V.

Note the big red border here

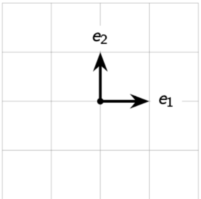
Important

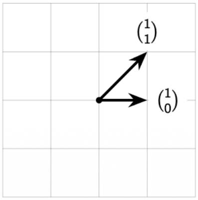
A subspace has *many different* bases, but they all have the same number of vectors.

Bases of $\mathbf{R}^2$

Question

What is a basis for $\mathbf{R}^2$?





Bases of $\mathbf{R}^n$

The unit coordinate vectors

$$e_1 = \begin{pmatrix}1\\0\\\vdots\\0\\0\end{pmatrix}, \quad e_2 = \begin{pmatrix}0\\1\\\vdots\\0\\0\end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix}0\\0\\\vdots\\1\\0\end{pmatrix}, \quad e_n = \begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}$$

are a basis for $\mathbf{R}^n$. (The identity matrix has columns $e_1, e_2, \ldots, e_n$.)

1. They span: I_n has a pivot in every row.
2. They are linearly independent: I_n has a pivot in every column.

In general: $\{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbf{R}^n$ if and only if the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

has a pivot in every row and every column.

Basis of a Subspace

Example

Example

Let

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \mid x + 3y + z = 0 \right\} \qquad \mathcal{B} = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}.$$

Verify that $\mathcal{B}$ is a basis for V. (So $\dim V = 2$: it is a *plane*.)

https://textbooks.math.gatech.edu/ila/demos/spans.html?v1=-3,1,0&v2=0,1,-3&range=5&captions=combo

Example: Find a basis for Nul(A) with A given below (the RREF of A is also given)

Basis for Nul A

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Fact

The vectors in the parametric vector form of the general solution to $Ax = 0$ always form a basis for Nul A.

Example: find a basis for Col(A) for the matrix A given below (the RREF of A is also given)

Basis for Col A

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns = basis ⟵ pivot columns in rref

Fact

The *pivot columns* of A always form a basis for Col A.

Warning: I mean the pivot columns of the *original* matrix A, not the row-reduced form. (Row reduction changes the column space.)

The Basis Theorem

Basis Theorem

Let V be a subspace of dimension m. Then:

- Any m linearly independent vectors in V form a basis for V.
- Any m vectors that span V form a basis for V.

Example: any three linearly independent vectors form a basis for $\mathbf{R}^3$.

Upshot

If you *already* know that $\dim V = m$, and you have m vectors $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ in V, then you only have to check *one* of

1. $\mathcal{B}$ is linearly independent, *or*
2. $\mathcal{B}$ spans V

in order for $\mathcal{B}$ to be a basis.

The Rank Theorem

Recall:

- The **dimension** of a subspace V is the number of vectors in a basis for V.
- A basis for the column space of a matrix A is given by the pivot columns.
- A basis for the null space of A is given by the vectors attached to the free variables in the parametric vector form.

Definition

The **rank** of a matrix A, written rank A, is the dimension of the column space Col A. The **nullity** of A, written nullity A, is the dimension of the solution set of $Ax = 0$.

Rank Theorem

If A is an $m \times n$ matrix, then

$$\operatorname{rank} A + \operatorname{nullity} A = n = \text{the number of columns of } A.$$

so that is:

(dimension of column space) + (dimension of solution set to Ax=0) = (number of variables)

The Rank Theorem

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

basis of Col A free variables

You Try It!

Poll

Poll

True or False: If A is a 10×15 matrix and there is a basis of Col A consisting of 4 vectors, then there is a basis of Nul A consisting of 6 vectors.

Summary

- A **basis** of a subspace is a minimal set of spanning vectors.
- There are recipes for computing a basis for the column space and null space of a matrix.
- The **dimension** of a subspace is the number of vectors in any basis.
- The **basis theorem** says that if you already know that $\dim V = m$, and you have m vectors in V, then you only have to check if they span *or* they're linearly independent to know they're a basis.
- The **rank theorem** says the dimension of the column space of a matrix, plus the dimension of the null space, is the number of columns of the matrix.