Chapter 3

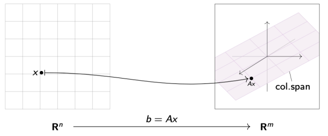
Linear Transformations and Matrix Algebra

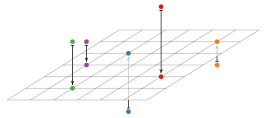
Section 3.1

Matrix Transformations

Motivation

Let A be a matrix, and consider the matrix equation $b = Ax$. If we vary x, we can think of this as a *function* of x.





[interactive]

Example: A=[1 0 ; 0 1 ; 0 0], plot some inputs x and outputs Ax

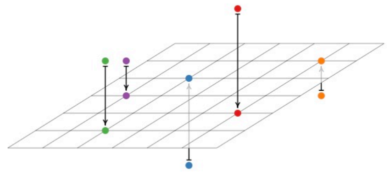
Example: with A as below, plot some inputs and outputs

Matrices as Functions

Projection

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?mat=1,0,0:0,1,0:0,0,0&range2=5&closed=true



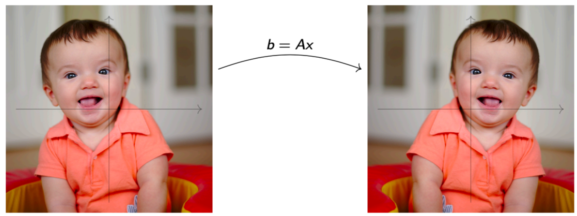
[interactive]

Example: for the matrix A given below, plot some input and output vectors for T(x)=Ax, and give the function T an appropriate NAME.

Matrices as Functions
Reflection

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

https://textbooks.math.gatech.edu/ila/demos/twobytwo.html?mat=-1,0,0,1&closed=true



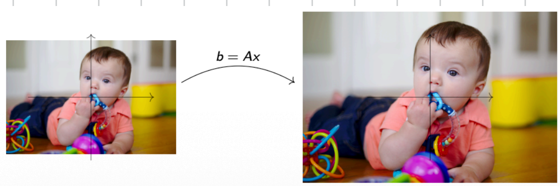
[interactive]

Example: plot some input and output vectors for T(x)=Ax and give the function T a NAME.

Matrices as Functions
Dilation

$$A = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}$$

https://textbooks.math.gatech.edu/ila/demos/twobytwo.html?mat=1.5,0,0,1.5&closed=true&pic=theo3.jpg



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Example: ditto

Matrices as Functions

Identity

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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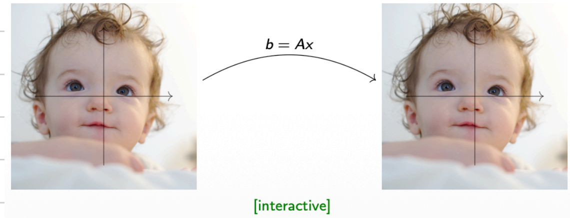
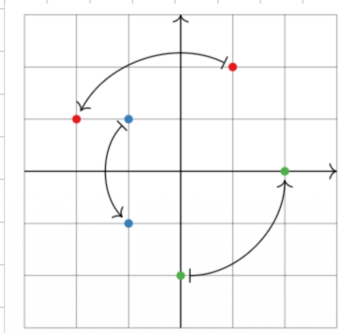


Matrices as Functions

Rotation

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



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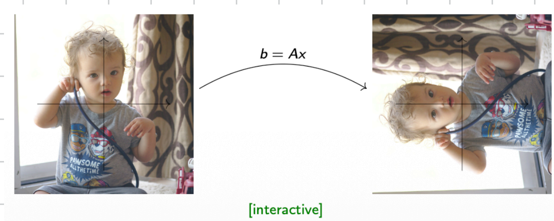


Other Geometric Transformations

In §3.1 there are other examples of geometric transformations of $\mathbf{R}^2$ given by matrices. Please look them over.

Transformations
Motivation

We have been drawing pictures of what it looks like to multiply a matrix by a vector, as a function of the vector.

Now let's go the other direction. Suppose we have a function, and we want to know, does it come from a matrix?

Example

For a vector x in $\mathbf{R}^2$, let $T(x)$ be the counterclockwise rotation of x by an angle θ. Is $T(x) = Ax$ for some matrix A?

If $\theta = 90^\circ$, then we know $T(x) = Ax$, where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

But for general θ, it's not clear.

Our next goal is to answer this kind of question.

Transformations
Vocabulary

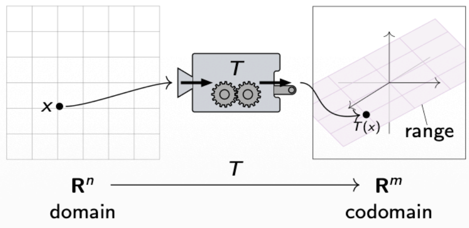
Definition

A **transformation** (or **function** or **map**) from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule T that assigns to each vector x in $\mathbf{R}^n$ a vector $T(x)$ in $\mathbf{R}^m$.

- $\mathbf{R}^n$ is called the **domain** of T (the inputs).
- $\mathbf{R}^m$ is called the **codomain** of T (where the outputs live).
- For x in $\mathbf{R}^n$, the vector $T(x)$ in $\mathbf{R}^m$ is the **image** of x under T.
  Notation: $x \mapsto T(x)$.
- The set of all images $\{T(x) \mid x \text{ in } \mathbf{R}^n\}$ is the **range** of T.

Notation:

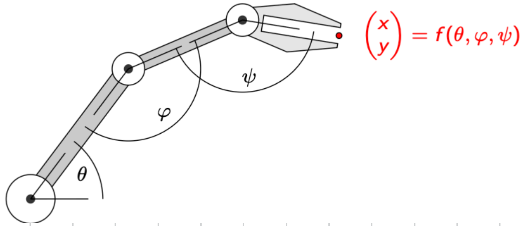
$T\colon \mathbf{R}^n \longrightarrow \mathbf{R}^m$ means T is a transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$.



Examples from calculus: f(x)=sin(x) and g(x)=x^2

Functions from Engineering

Suppose you are building a robot arm with three joints that can move its hand around a plane, as in the following picture.



Matrix Transformations

Definition

Let A be an $m \times n$ matrix. The **matrix transformation** associated to A is the transformation

$$T : \mathbf{R}^n \longrightarrow \mathbf{R}^m \quad \text{defined by} \quad T(x) = Ax.$$

Example: find the domain, codomain, and range of the matrix transformation T(x) below

$$T\begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -14 \\ -32 \end{pmatrix}.$$

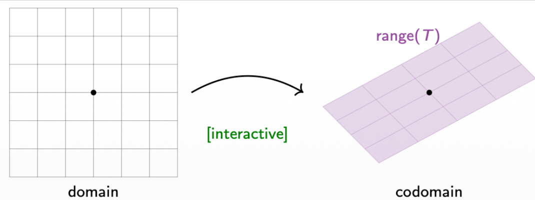
- The *domain* of T is $\mathbf{R}^n$, which is the number of *columns* of A.
- The *codomain* of T is $\mathbf{R}^m$, which is the number of *rows* of A.
- The *range* of T is the set of all images of T:

$$T(x) = Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

Example: find the domain, codomain, and range of the matrix transformation T(x)=Ax with the matrix A given below

$$A = \begin{pmatrix} -1 & 0 \\ 2 & 1 \\ 1 & -1 \end{pmatrix}$$

https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?mat=-1,0:2,1:1,-1&closed=true&show=false



You Try It!

Poll

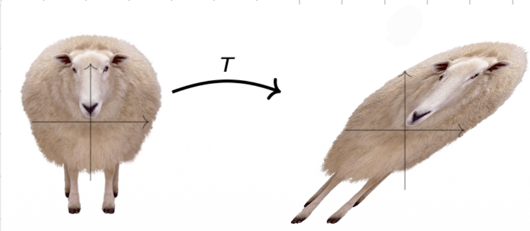
Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T: \mathbf{R}^2 \to \mathbf{R}^2$. ($T$ is called a **shear**.)

Find: the domain, the codomain, and the range of T.

Also plot a few input and output vectors, and give the transformation a NAME

Summary

- We can think of $b = Ax$ as a **transformation** with input x and output b.
- There are vocabulary words associated to transformations: **domain**, **codomain**, **range**.
- A transformation that comes from a matrix is called a **matrix transformation**.
- In this case, the vocabulary words mean something concrete in terms of matrices.
- We like transformations that come from matrices, because questions about those transformations turn into questions about matrices.



Section 3.2

One-to-one and Onto Transformations

Matrix Transformations
Reminder

Recall: Let A be an $m \times n$ matrix. The **matrix transformation** associated to A is the transformation

$$T\colon \mathbf{R}^n \longrightarrow \mathbf{R}^m \quad \text{defined by} \quad T(x) = Ax.$$

- The *domain* of T is $\mathbf{R}^n$, which is the number of *columns* of A.
- The *codomain* of T is $\mathbf{R}^m$, which is the number of *rows* of A.
- The *range* of T is the set of all images of T:

$$T(x) = Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

This is the *column space* of A. It is a span of vectors in the codomain.

Matrix Transformations
Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^2 \to \mathbf{R}^3$.

(1) find T(u) where u=[3;4]

(2) let b=[7;5;7], find v in R^2 such that T(v)=b.

(3) is there any other vector w (other than v) that also gets sent to b?

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T : \mathbf{R}^2 \to \mathbf{R}^3$.

Same A as previous example.

(4) is there any vector c in R^3 which is not in the range of T? In other words, can you find a vector c for which NO vector v in R^2 maps to c?

Translation: Find c such that $Ax = c$ is inconsistent.

Matrix Transformations
Non-Example

Note: All of these questions are questions about *the transformation* T; it still makes sense to ask them in the absence of the matrix A.

The fact that T comes from a matrix means that these questions translate into questions about a matrix, which we know how to do.

Non-example: $T: \mathbf{R}^2 \to \mathbf{R}^3$ $\quad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin x \\ xy \\ \cos y \end{pmatrix}$

We will restrict our focus to matrix transformations in this class, but it is good to know that you can still ask similar questions about arbitrary transformations from R^n to R^m.

Questions About Transformations

Today we will focus on two important questions one can ask about a transformation $T: \mathbf{R}^n \to \mathbf{R}^m$:

- Do there exist distinct vectors x, y in $\mathbf{R}^n$ such that $T(x) = T(y)$?
- For every vector v in $\mathbf{R}^m$, does there exist a vector x in $\mathbf{R}^n$ such that $T(x) = v$?

These are subtle because of the multiple *quantifiers* involved ("for every", "there exists").

One-to-one Transformations

Definition

A transformation $T: \mathbf{R}^n \to \mathbf{R}^m$ is **one-to-one** (or **into**, or **injective**) if different vectors in $\mathbf{R}^n$ map to different vectors in $\mathbf{R}^m$. In other words, for every b in $\mathbf{R}^m$, the equation $T(x) = b$ has *at most one* solution x. Or, different inputs have different outputs. Note that *not* one-to-one means at least two different vectors in $\mathbf{R}^n$ have the same image.

Matrix Transformations
Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T: \mathbf{R}^2 \to \mathbf{R}^3$.

Same example as before: Is the transformation T(x)=Ax one-to-one?

Characterization of One-to-One Matrix Transformations

Theorem

Let $T: \mathbf{R}^n \to \mathbf{R}^m$ be a matrix transformation with matrix A. Then the following are equivalent:

- T is one-to-one.
- For each b in $\mathbf{R}^m$, the equation $T(x) = b$ has at most one solution.
- For each b in $\mathbf{R}^m$, the equation $Ax = b$ has a unique solution or is inconsistent.
- $Ax = 0$ has a unique solution.
- The columns of A are linearly independent.
- A has a pivot in every column.

Question

If $T: \mathbf{R}^n \to \mathbf{R}^m$ is one-to-one, what can we say about the relative sizes of n and m?

New example:

One-to-One Transformations
Example

Define

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad T(x) = Ax,$$

so $T : \mathbf{R}^2 \to \mathbf{R}^3$. Is T one-to-one?

Non-example:

One-to-One Transformations
Non-Example

Define

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad T(x) = Ax,$$

(1) Is the transformation T(x)=Ax one-to-one?

(2) If not, find two **different** vectors v and w that map to the same output b

(3) Also, is **every** vector b in R^2 the output for some input vector v?

Onto Transformations

Definition

A transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ is **onto** (or **surjective**) if the range of T is equal to $\mathbf{R}^m$ (its codomain). In other words, for every b in $\mathbf{R}^m$, the equation $T(x) = b$ has at *least one solution*. Or, every possible output has an input. Note that *not* onto means there is some b in $\mathbf{R}^m$ which is not the image of any x in $\mathbf{R}^n$.

Characterization of Onto Matrix Transformations

Theorem

Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a matrix transformation with matrix A. Then the following are equivalent:

- T is onto
- $T(x) = b$ has a solution for every b in $\mathbf{R}^m$
- $Ax = b$ is consistent for every b in $\mathbf{R}^m$
- The columns of A span $\mathbf{R}^m$
- A has a pivot in every row

Same example as before:

One-to-One Transformations

Non-Example

Define

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad T(x) = Ax,$$

(1) is T(x)=Ax onto?

Same example as before:

Matrix Transformations

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T : \mathbf{R}^2 \to \mathbf{R}^3$.

(1) is T(x)=Ax onto?

Question

If $T : \mathbf{R}^n \to \mathbf{R}^m$ is onto, what can we say about the relative sizes of n and m?

Last example:

One-to-One and Onto Transformations
Non-Example

Define
$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \qquad T(x) = Ax,$$
so $T\colon \mathbf{R}^3 \to \mathbf{R}^2$. Is T one-to-one? Is it onto?

(1) find the domain and codomain of T

(2) is T one-to-one?

(3) is T onto?

Summary

- A transformation T is **one-to-one** if $T(x) = b$ has *at most one* solution, for every b in $\mathbf{R}^m$.
- A transformation T is **onto** if $T(x) = b$ has *at least one* solution, for every b in $\mathbf{R}^m$.
- A matrix transformation with matrix A is one-to-one if and only if the columns of A are linearly independent, if and only if A has a pivot in every column.
- A matrix transformation with matrix A is onto if and only if the columns of A span $\mathbf{R}^m$, if and only if A has a pivot in every row.
- Two of the most basic questions one can ask about a transformation is whether it is one-to-one or onto.

Section 3.3

Linear Transformations

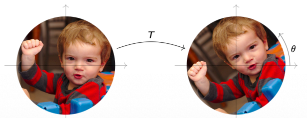
Linear Transformations
Motivation

In the last two lectures we have been asking questions about transformations, and answering them in the case of matrix transformations.

However, sometimes it is not clear if a transformation is a matrix transformation or not.

Example

For a vector x in $\mathbf{R}^2$, let $T(x)$ be the counterclockwise rotation of x by an angle θ. Is $T(x) = Ax$ for some matrix A?



Key observation:

Recall: If A is a matrix, u, v are vectors, and c is a scalar, then

$$A(u+v) = Au + Av \qquad A(cv) = cAv.$$

So if $T(x) = Ax$ is a matrix transformation then,

$$T(u+v) = T(u) + T(v) \quad \text{and} \quad T(cv) = cT(v).$$

Linear Transformations

Definition

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is **linear** if it satisfies the equations for all vectors u, v in $\mathbf{R}^n$ and all scalars c.

(1) T(u+v) = T(u) + T(v) "T respects addition"

(2) T(cv) = c T(v) "T respects scalar multiplication"

Some immediate consequences:

Check: if T is linear, then

$$T(0) = 0 \qquad T(cu + dv) = cT(u) + dT(v)$$

for all vectors u, v and scalars c, d. More generally,

$$T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n).$$

In engineering this is called **superposition**.

Linear Transformations
Dilation

Define $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = 1.5x$. Is T linear? Check:

Note: T is a matrix transformation!

$$T(x) = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix} x,$$

as we checked before.

Linear Transformations
Rotation

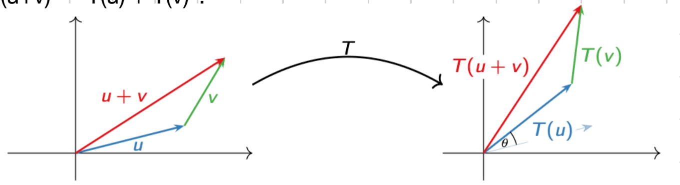
Define $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ by

$T(x) =$ the vector x rotated counterclockwise by an angle of θ.

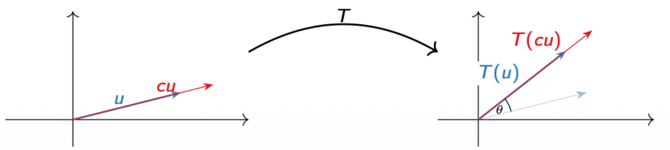
Is T linear? Check:

Proof by picture:

$T(u+v) \stackrel{?}{=} T(u) + T(v)$?



$T(cv) \stackrel{?}{=} c\,T(v)$?



Linear Transformations
Non-example

Is every transformation a linear transformation?

No! For instance, $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin x \\ xy \\ \cos y \end{pmatrix}$ is not linear.

Hint: check what happens to T(v) if v=0 the zero vector.

You Try It!

Poll

Which of the following transformations are linear?

A. $T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}|x_1|\\x_2\end{pmatrix}$ **B.** $T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}2x_1 + x_2\\x_1 - 2x_2\end{pmatrix}$

C. $T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1x_2\\x_2\end{pmatrix}$ **D.** $T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}2x_1 + 1\\x_1 - 2x_2\end{pmatrix}$

The Matrix of a Linear Transformation

We will see that a *linear* transformation T is a matrix transformation: $T(x) = Ax$.

But what matrix does T come from? What is A?

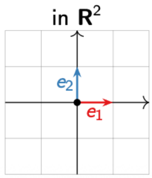
Here's how to compute it.

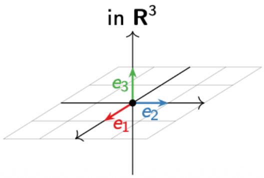
The Matrix of a Linear Transformation

We will see that a *linear* transformation T is a matrix transformation: $T(x) = Ax$.

But what matrix does T come from? What is A?

Here's how to compute it.





First: recall the definitions of e1,e2,...,en

Unit Coordinate Vectors

Definition

The **unit coordinate vectors** in $\mathbf{R}^n$ are

This is what $e_1, e_2, \ldots$ mean, for the rest of the class.

$$e_1 = \begin{pmatrix}1\\0\\ \vdots\\0\\0\end{pmatrix}, \quad e_2 = \begin{pmatrix}0\\1\\ \vdots\\0\\0\end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix}0\\0\\ \vdots\\1\\0\end{pmatrix}, \quad e_n = \begin{pmatrix}0\\0\\ \vdots\\0\\1\end{pmatrix}.$$

For the matrix A=[1 2 3 ; 4 5 6 ; 7 8 9] compute Ae1, Ae2, and Ae3.

$$\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

Note: if A is an $m \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, then $Ae_i = v_i$ for $i = 1, 2, \ldots, n$: multiplying a matrix by e_i gives you the ith column.

Linear Transformations are Matrix Transformations

Theorem

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Let

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}.$$

This is an $m \times n$ matrix, and T is the matrix transformation for A: $T(x) = Ax$.

The matrix A is called the **standard matrix** for T.

Take-Away

Linear transformations are the same as matrix transformations.

Linear Transformations are Matrix Transformations
Example

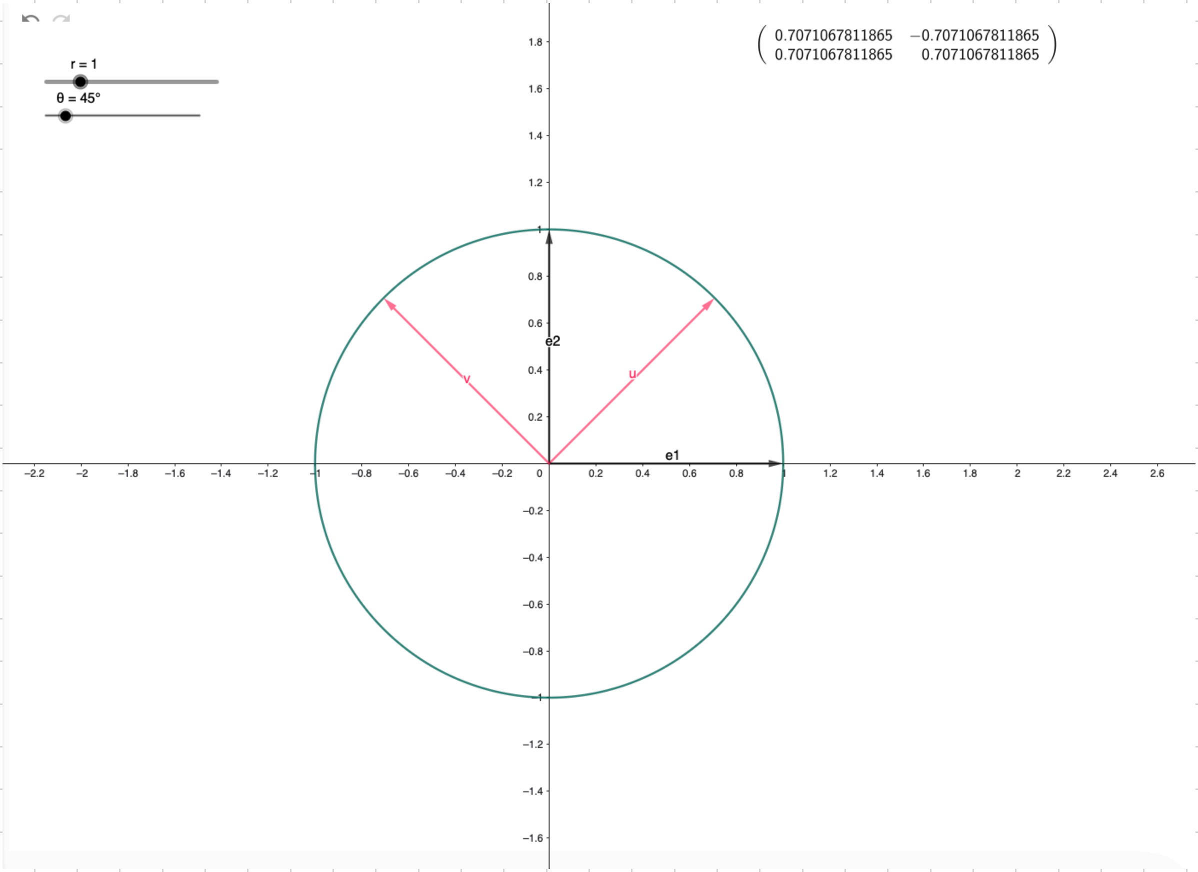
Before, we defined a **dilation** transformation $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = 1.5x$.
What is its standard matrix?

Linear Transformations are Matrix Transformations
Example

Question
What is the matrix for the linear transformation $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$T(x) = x \text{ rotated counterclockwise by an angle } \theta?$$

https://www.geogebra.org/graphing/jyzexjma



Linear Transformations are Matrix Transformations
Example, continued

Question

What is the matrix for the linear transformation $T: \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?

(1) Find the standard matrix A of T

(2) is T one-to-one?

(3) is T onto?

Linear Transformations are Matrix Transformations
Example

Question

Define a linear transformation $T: \mathbf{R}^3 \to \mathbf{R}^2$ by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ -y - 5z \end{pmatrix}.$$

What is the standard matrix A for T?

(1) Find the standard matrix A of T

(2) is T one-to-one?

(3) is T onto?

Summary

- **Linear transformations** are the transformations that come from matrices.
- The **unit coordinate vectors** $e_1, e_2, \ldots$ are the unit vectors in the positive direction along the coordinate axes.
- You compute the columns of the matrix for a linear transformation by plugging in the unit coordinate vectors.
- This is useful when the transformation is specified geometrically, in terms of a formula, or any other way that isn't as a matrix transformation.