Section 3.4

Matrix Multiplication

Motivation

Recall: we can turn any system of linear equations into a matrix equation

$$Ax = b.$$

This notation is suggestive. Can we solve the equation by "dividing by A"?

$$x \overset{??}{=} \frac{b}{A}$$

Answer: Sometimes, but you have to know what you're doing.

Today we'll study *matrix algebra*: adding and multiplying matrices.

These are not so hard to do. The important thing to understand today is the relationship between *matrix multiplication* and *composition of transformations*.

More Notation for Matrices

Let A be an $m \times n$ matrix.

We write a_{ij} for the entry in the ith row and the jth column. It is called the **ijth entry** of the matrix.

$$\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}$$

ith row

jth column

The entries $a_{11}, a_{22}, a_{33}, \ldots$ are the **diagonal entries**; they form the **main diagonal** of the matrix.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \qquad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

A **diagonal matrix** is a *square* matrix whose only nonzero entries are on the main diagonal.

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

The $n \times n$ **identity matrix** I_n is the diagonal matrix with all diagonal entries equal to 1. It is special because $I_n v = v$ for *all* v in $\mathbf{R}^n$.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

More Notation for Matrices
Continued

The **zero matrix** (of size $m \times n$) is the $m \times n$ matrix 0 with all zero entries.

$$0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose rows are the columns of A. In other words, the ij entry of A^T is a_{ji}.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \rightsquigarrow A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}$$

flip

Matrix Multiplication

must be equal

Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix with columns $v_1, v_2, \ldots, v_p$:

$$B = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

The **product** AB is the $m \times p$ matrix with columns $Av_1, Av_2, \ldots, Av_p$:

The equality is a definition

$$AB \overset{\text{def}}{=} \begin{pmatrix} | & | & & | \\ Av_1 & Av_2 & \cdots & Av_p \\ | & | & & | \end{pmatrix}.$$

In order for $Av_1, Av_2, \ldots, Av_p$ to make sense, the number of columns of A has to be the same as the number of rows of B. Note the sizes of the product!

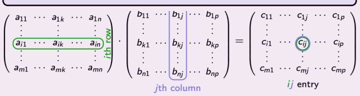
Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \left(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 14 & -10 \\ 32 & -28 \end{pmatrix}$$

The ij entry of $C = AB$ is the ith row of A times the jth column of B:

$$c_{ij} = (AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

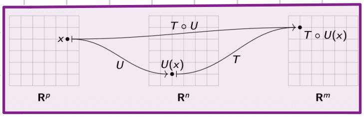


The Row-Column Rule for Matrix Multiplication

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1\cdot1+2\cdot2+3\cdot3 & \square \\ \square & \square \end{pmatrix} = \begin{pmatrix} 14 & \square \\ \square & \square \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} \square & \square \\ 4\cdot1+5\cdot2+6\cdot3 & \square \end{pmatrix} = \begin{pmatrix} \square & \square \\ 32 & \square \end{pmatrix}$$



The matrix of the composition is the product of the matrices!

Composition of Transformations

Definition

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ and $U\colon \mathbf{R}^p \to \mathbf{R}^n$ be transformations. The **composition** is the transformation

$$T \circ U\colon \mathbf{R}^p \to \mathbf{R}^m \quad \text{defined by} \quad T \circ U(x) = T(U(x)).$$

Example: Find the standard matrix for the transformation T ○U where
U is the transformation which rotates vectors in R^2 by 90^deg *clockwise*, and
T is the transformation which projects vectors in R^2 to the x-axis.

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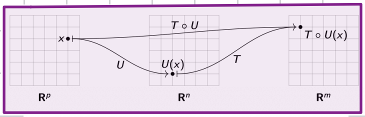

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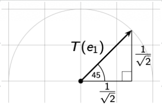
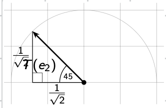
Composition of Linear Transformations

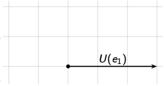
Another Example

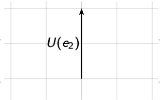
Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be rotation by 45°, and let $U: \mathbf{R}^2 \to \mathbf{R}^2$ scale the x-coordinate by 1.5. Let's compute their standard matrices A and B:

Also: find the standard matrix for $C = T \circ U$









You Try It!

You Try It!

Poll

Poll

Do there exist *nonzero* matrices A and B with $AB = 0$?

Hint: think geometrically!

You Try It!

Composition of Linear Transformations
Example

Let $T\colon \mathbf{R}^3 \to \mathbf{R}^2$ and $U\colon \mathbf{R}^2 \to \mathbf{R}^3$ be the matrix transformations

$$T(x) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} x \qquad U(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} x.$$

Example: What is the standard matrix for the transformation which first applies the transformation T to a vector x and then applies the transformation U to the result?

Addition and Scalar Multiplication for Matrices

You add two matrices component by component, like with vectors.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{pmatrix}$$

Note you can only add two matrices *of the same size.*

You multiply a matrix by a scalar by multiplying each component, like with vectors.

$$c\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \end{pmatrix}.$$

These satisfy the expected rules, like with vectors:

$$A+B = B+A \qquad (A+B)+C = A+(B+C)$$
$$c(A+B) = cA+cB \qquad (c+d)A = cA+dA$$
$$(cd)A = c(dA) \qquad A+0 = A$$

Properties of Matrix Multiplication

Mostly matrix multiplication works like you'd expect. Suppose A has size $m \times n$, and that the other matrices below have the right size to make multiplication work.

$$A(BC) = (AB)C \qquad A(B+C) = (AB+AC)$$
$$(B+C)A = BA+CA \qquad c(AB) = (cA)B$$
$$c(AB) = A(cB) \qquad I_m A = A$$
$$AI_n = A$$

Warnings!

- AB is usually not equal to BA.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} =$$

Warnings!

- $AB = AC$ does not imply $B = C$, even if $A \neq 0$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix} =$$

Warnings!

- $AB = 0$ does not imply $A = 0$ or $B = 0$.

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} =$$

Powers of a Matrix

Definition

Let n be a positive whole number and let A be a square matrix. The n**th power** of A is the product

$$A^n = \underbrace{A \cdot A \cdots A}_{n \text{ times}}$$

Example

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$A^2 =$$

$$A^3 =$$

Summary

- The product of an $m \times n$ matrix and an $n \times p$ matrix is an $m \times p$ matrix. I showed you two ways of computing the product.
- Composition of linear transformations corresponds to multiplication of matrices.
- You have to be careful when multiplying matrices together, because things like commutativity and cancellation fail.
- You can take powers of square matrices.

Section 3.5 and 3.6

Matrix Inverses and the Invertible Matrix Theorem

The Definition of Inverse

Definition

Let A be an $n \times n$ square matrix. We say A is **invertible** (or **nonsingular**) if there is a matrix B of the same size, such that

$$AB = I_n \quad \text{and} \quad BA = I_n.$$

identity matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

In this case, B is the **inverse** of A, and is written A^{-1}.

Example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

I claim $B = A^{-1}$. Check:

You Try It!

Poll

Do there exist two matrices A and B such that AB is the identity, but BA is not? If so, find an example. (Both products have to make sense.)

Hint: don't assume A and B have the same size.

However

If A and B are *square* matrices, then

$$AB = I_n \quad \text{if and only if} \quad BA = I_n.$$

So in this case you only have to check one.

Key observation: Having an inverse of a coefficient matrix of a system Ax=b makes solving the system very easy!

Solving Linear Systems via Inverses
Example

Example

Solve the system

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 &= 1 \\ x_1 \quad\quad + 3x_3 &= 1 \\ 2x_1 + 2x_2 + 3x_3 &= 1 \end{aligned} \quad \text{using} \quad \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}.$$

The advantage of using inverses is it doesn't matter what's on the right-hand side of the $=$:

$$\begin{cases} 2x_1 + 3x_2 + 2x_3 = b_1 \\ x_1 \quad\quad + 3x_3 = b_2 \\ 2x_1 + 2x_2 + 3x_3 = b_3 \end{cases} \implies \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$= \begin{pmatrix} -6b_1 - 5b_2 + 9b_3 \\ 3b_1 + 2b_2 - 4b_3 \\ 2b_1 + 2b_2 - 3b_3 \end{pmatrix}$$

Important

If A is invertible and you know its inverse, then the easiest way to solve $Ax = b$ is by "dividing by A":

$$x = A^{-1}b.$$

This is very convenient when you have to vary b!

In particular, we have the following:

FACT: If A is an invertible square matrix, then Ax=b is <u>**always consistent**</u> and there is exactly one <u>**unique solution**</u>.

Computing A^{-1}

The 2×2 case

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The **determinant** of A is the number

$$\det(A) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Facts:

1. If $\det(A) \neq 0$, then A is invertible and $A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
2. If $\det(A) = 0$, then A is not invertible.

Example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} =$$

Invertible Linear Transformations

Examples

Let T = counterclockwise rotation in the plane by 45°. Its matrix is

$$A = \begin{pmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then T^{-1} = counterclockwise rotation by -45°. Its matrix is what matrix?

Let T = counterclockwise rotation in the plane by 45°. What is T^{-1}?

Invertible Transformations

Definition

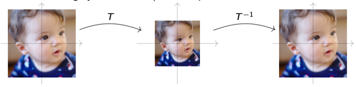
A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^n$ is **invertible** if there exists another transformation $U\colon \mathbf{R}^n \to \mathbf{R}^n$ such that

$$T \circ U(x) = x \quad \text{and} \quad U \circ T(x) = x$$

for all x in $\mathbf{R}^n$. In this case we say U is the **inverse** of T, and we write $U = T^{-1}$.

In other words, $T(U(x)) = x$, so T "undoes" U, and likewise U "undoes" T.

Let T = shrinking by a factor of 2/3 in the plane. What is T^{-1}?



Fact

If T is an invertible linear transformation with matrix A, then T^{-1} is an invertible linear transformation with matrix A^{-1}.

Fact

A transformation T is invertible if and only if it is both one-to-one and onto.

Some Facts

Say A and B are invertible $n \times n$ matrices.

1. A^{-1} is invertible and its inverse is $(A^{-1})^{-1} = A$.
2. AB is invertible and its inverse is $(AB)^{-1} =$ ~~$A^{-1}B^{-1}$~~ $B^{-1}A^{-1}$.

   Why? $(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$.
3. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

   Why? $A^T(A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n$.

You Try It!

Question: If A, B, C are invertible $n \times n$ matrices, what is the inverse of ABC?

Computing A^{-1}

In general

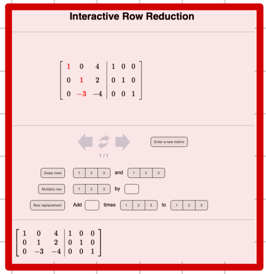
Let A be an $n \times n$ matrix. Here's how to compute A^{-1}.

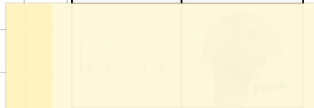
1. Row reduce the augmented matrix $(\, A \mid I_n \,)$.
2. If the result has the form $(\, I_n \mid B \,)$, then A is invertible and $B = A^{-1}$.
3. Otherwise, A is not invertible.

Example

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}$$

https://textbooks.math.gatech.edu/ila/demos/rrinter.html?mat=1,0,4,1,0,0:0,1,2,0,1,0:0,-3,-4,0,0,1&augment=2





The Invertible Matrix Theorem

Let A be an $n \times n$ matrix, and let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. A is invertible.
2. T is invertible.
3. The reduced row echelon form of A is the identity matrix I_n.
4. A has n pivots.
5. $Ax = 0$ has no solutions other than the trivial solution.
6. $\text{Nul}(A) = \{0\}$.
7. $\text{nullity}(A) = 0$.
8. The columns of A are linearly independent.
9. The columns of A form a basis for $\mathbf{R}^n$.
10. T is one-to-one.
11. $Ax = b$ is consistent for all b in $\mathbf{R}^n$.
12. $Ax = b$ has a unique solution for each b in $\mathbf{R}^n$.
13. The columns of A span $\mathbf{R}^n$.
14. $\text{Col}\, A = \mathbf{R}^n$.
15. $\dim \text{Col}\, A = n$.
16. $\text{rank}\, A = n$.
17. T is onto.
18. There exists a matrix B such that $AB = I_n$.
19. There exists a matrix $BA = I_n$ such that $BA = I_n$.

you really have to know these

The Invertible Matrix Theorem
Summary

There are two kinds of *square* matrices:

1. invertible (non-singular), and
2. non-invertible (singular).

For invertible matrices, all statements of the Invertible Matrix Theorem are true.

For non-invertible matrices, all statements of the Invertible Matrix Theorem are false.

The Invertible Matrix Theorem
Example

Question: Is this matrix invertible?

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & 7 \\ -2 & -4 & 1 \end{pmatrix}$$

You Try It!

The Invertible Matrix Theorem
Another Example

Problem: Let A be a 3×3 matrix such that

$$A\begin{pmatrix} 1 \\ 7 \\ 0 \end{pmatrix} = A\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}.$$

True or False: the rank of A is equal to 3.

Summary

- The **inverse** of a square matrix A is a matrix A^{-1} such that $AA^{-1} = I_n$ (equivalently, $A^{-1}A = I_n$).
- If A is invertible, then you can solve $Ax = b$ by "dividing by A": $b = A^{-1}x$. There is a unique solution $x = A^{-1}b$ for every x.
- You compute A^{-1} (and whether A is invertible) by row reducing $\left(A \mid I_n \right)$. There's a trick for computing the inverse of a 2×2 matrix in terms of determinants.
- A linear transformation T is invertible if and only if its matrix A is invertible, in which case A^{-1} is the matrix for T^{-1}.
- The Invertible Matrix theorem is a list of a zillion equivalent conditions for invertibility that you have to learn (and should understand, since it's well within what we've covered in class so far).