Chapter 4

Determinants

Section 4.1

Determinants: Definition

Orientation

Recall: This course is about learning to:

- Solve the matrix equation $Ax = b$
  We've said most of what we'll say about this topic now.
- Solve the matrix equation $Ax = \lambda x$ (eigenvalue problem)
  We are now aiming at this.
- Almost solve the equation $Ax = b$
  This will happen later.

Today is mostly about the *theory* of the determinant; in the next lecture we will focus on *computation*.

A Definition of Determinant

Definition
The **determinant** is a function

determinants are only for square matrices!

$$\det\colon \{n \times n \text{ matrices}\} \longrightarrow \mathbf{R}$$

with the following properties:

1. If you do a row replacement on a matrix, the determinant doesn't change.
2. If you scale a row by c, the determinant is multiplied by c.
3. If you swap two rows of a matrix, the determinant is multiplied by -1.
4. $\det(I_n) = 1$.

This is a *definition* because it tells you how to compute the determinant: row reduce!

Example:

$$\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$$

Special Cases

Special Case 1
If A has a zero row, then $\det(A) = 0$.

Special Cases

Special Case 2
If A is upper-triangular, then the determinant is the product of the diagonal entries:

$$\det\begin{pmatrix} a & \star & \star \\ 0 & b & \star \\ 0 & 0 & c \end{pmatrix} = abc.$$

Example:

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -10 \\ 0 & 0 & -74 \end{pmatrix}$$

Compute the determinant of the matrix.

$$\begin{pmatrix} 0 & -7 & -4 \\ 2 & 4 & 6 \\ 3 & 7 & -1 \end{pmatrix}$$

Computing Determinants
2×2 Example

Let's compute the determinant of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, a general 2×2 matrix.

You try it!

Example: $\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Example: $\det \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 3 & 4 \end{pmatrix}$

Poll

Poll

True or false:

(a) Row operations can change the determinant of a matrix.

(b) Row operations can change whether the determinant of a matrix is equal to zero.

Determinants and Invertibility

Theorem
A square matrix A is invertible if and only if $\det(A)$ is nonzero.

Why?

- If A is invertible, then its reduced row echelon form is the identity matrix, which has determinant equal to 1.
- If A is not invertible, then its reduced row echelon form has a zero row, hence has zero determinant.
- Doing row operations doesn't change whether the determinant is zero.

Example: Is the matrix invertible? $\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}$

Example: Is the matrix invertible? $\begin{pmatrix} 3 & 2 \\ 6 & 2 \end{pmatrix}$

Example: Is the matrix invertible? $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

Determinants and Products

Theorem
If A and B are two $n \times n$ matrices, then

$$\det(AB) = \det(A) \cdot \det(B).$$

Example: Compute the determinant of the matrix. $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 6 & 2 \end{pmatrix}$

Example: Compute the determinant of the matrix. $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}$

Transposes
Review

Recall: The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose rows are the columns of A. In other words, the ij entry of A^T is a_{ji}.

$$\overset{A}{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}} \rightsquigarrow \overset{A^T}{\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}}$$

flip

Determinants and Transposes

Theorem
If A is a square matrix, then

$$\det(A) = \det(A^T),$$

where A^T is the transpose of A.

Example: $\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \det\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$

As a consequence, det behaves the same way with respect to *column* operations as row operations.

Corollary ← an immediate consequence of a theorem
If A has a zero column, then $\det(A) = 0$.

Corollary
The determinant of a *lower*-triangular matrix is the product of the diagonal entries.

(The transpose of a lower-triangular matrix is upper-triangular.)

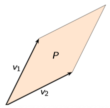
Example: Compute the determinant of the matrix. $\begin{pmatrix} 1 & 0 & 3 \\ 2 & 0 & 2 \\ 6 & 0 & 4 \end{pmatrix}$

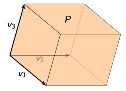
Section 4.3

Determinants and Volumes

Determinants and Volumes

The columns $v_1, v_2, \ldots, v_n$ of an $n \times n$ matrix A give you n vectors in $\mathbf{R}^n$. These determine a **parallelepiped** P.





Theorem

Let A be an $n \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, and let P be the parallelepiped determined by A. Then

$$(\text{volume of } P) = |\det(A)|.$$

Example: Find the volume of the parallelpiped determined by the vectors.

(a) $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$

(b) $\begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}0\\3\end{pmatrix}$

(c) $\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}$

(d) $\begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\3\end{pmatrix}$

Sanity check: the volume of P is zero $\iff$ the columns are *linearly dependent* (P is "flat") $\iff$ the matrix A is not invertible.

You try it!

Find (a) the determinant of the given matrix, (b) sketch the parallelpiped determined by the columns of the matrix, and (c) determine the volume of the parallelpiped.

$$\det\begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} =$$

$$\det\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} =$$

$$\det\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} =$$

Determinants and Volumes

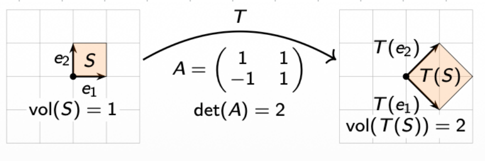
Theorem

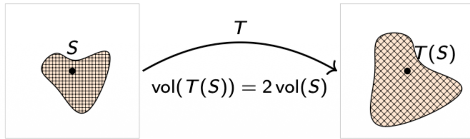
Let A be an $n \times n$ matrix, and let $T(x) = Ax$. If S is any region in $\mathbf{R}^n$, then

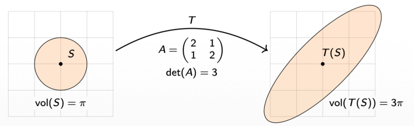
$$(\text{volume of } T(S)) = |\det(A)|\,(\text{volume of } S).$$

Example: Let S be the unit disk in $\mathbf{R}^2$, and let $T(x) = Ax$ for

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$







Summary

Magical Properties of the Determinant

you really have to know these

1. There is one and only one function $\det\colon \{\text{square matrices}\} \to \mathbf{R}$ satisfying the properties (1)–(4) on the second slide.
2. A is invertible if and only if $\det(A) \neq 0$.
3. The determinant of an upper- or lower-triangular matrix is the product of the diagonal entries.
4. If we row reduce A to row echelon form B using r swaps, then
$$\det(A) = (-1)^r \frac{(\text{product of the diagonal entries of } B)}{(\text{product of the scaling factors})}.$$
5. $\det(AB) = \det(A)\det(B)$ and $\det(A^{-1}) = \det(A)^{-1}$.
6. $\det(A) = \det(A^T)$.
7. $|\det(A)|$ is the volume of the parallelepiped defined by the columns of A.
8. If A is an $n \times n$ matrix with transformation $T(x) = Ax$, and S is a subset of $\mathbf{R}^n$, then the volume of $T(S)$ is $|\det(A)|$ times the volume of S. (Even for curvy shapes S.)

Section 4.2

Cofactor Expansions

Orientation

Last time: we learned...

- ...the definition of the determinant.
- ...to compute the determinant using row reduction.
- ...all sorts of magical properties of the determinant, like
  - $\det(AB) = \det(A)\det(B)$
  - the determinant computes volumes
  - nonzero determinants characterize invertability
  - etc.

Today: we will learn...

- Special formulas for 2×2 and 3×3 matrices.
- How to compute determinants using *cofactor expansions*.
- How to compute inverses using determinants.

Determinants of 2×2 Matrices

Reminder

We already have a formula in the 2×2 case:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Determinants of 3×3 Matrices

Here's the formula:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{array}{l} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ \quad - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \end{array}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Cofactor Expansions

Example

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\det\begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix} =$$

Cofactor Expansion
Example

Compute det(A), and notice the last column of mostly zeros.

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 5 & 9 & 1 \end{pmatrix}$$

expand along row i

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3}$$

expand down column j

$$\det(A) = a_{1j}c_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j}$$

You Try It!

Poll

Poll

$$\det \begin{pmatrix} 0 & 7 & 2 & 9 & 8 \\ 1 & 3 & 2 & 7 & 4 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 5 \end{pmatrix} = ?$$

A. -84 **B.** -28 **C.** -7 **D.** 0 **E.** 7 **F.** 28 **G.** 84

Cofactor Expansion

Advice

- In general, computing a determinant by cofactor expansion is slower than by row reduction.
- It makes sense to expand by cofactors if you have a row or column with a lot of zeros.
- Also if your matrix has unknowns in it, since those are hard to row reduce (you don't know where the pivots are).

You can also use more than one method; for example:

- Use cofactors on a 4×4 matrix but compute the minors using the 3×3 formula.
- Do row operations to produce a row/column with lots of zeros, then expand cofactors (but keep track of how you changed the determinant!).

Example:

$$\det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

Summary

We have several ways to compute the determinant of a matrix.

- Special formulas for 2×2 and 3×3 matrices.

  These work great for small matrices.
- Cofactor expansion.

  This is perfect when there is a row or column with a lot of zeros, or if your matrix has unknowns in it.
- Row reduction.

  This is the way to go when you have a big matrix which doesn't have a row or column with a lot of zeros.
- Any combination of the above.

  Cofactor expansion is recursive, but you don't have to use cofactor expansion to compute the determinants of the minors! Or you can do row operations and then a cofactor expansion.