Chapter 5

Eigenvalues and Eigenvectors

Section 5.1

Eigenvalues and Eigenvectors

Definition

If A is an $n \times n$ square matrix and there is a nonzero vector v in $\mathbf{R}^n$ and a scalar λ in $\mathbf{R}$ with

$$Av = \lambda v,$$

then v is an **eigenvector** of A with **eigenvalue** λ.

Note: Eigenvectors are by definition nonzero. Eigenvalues may be equal to zero.

Note: This is the most important definition in the course.

Verifying Eigenvectors

Example

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$$

Example

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

You Try It!

Poll

Poll

Which of the vectors

A. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ B. $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ C. $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ D. $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ E. $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

are eigenvectors of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$?

What are the eigenvalues?

Verifying Eigenvalues

Question: Is $\lambda = 3$ an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$?

Q: is there a **nonzero** vector x such that Ax=3x?

https://textbooks.math.gatech.edu/ila/demos/rabbits/book_demo.html

A Biology Question
Motivation

In a population of rabbits:

1. half of the newborn rabbits survive their first year;
2. of those, half survive their second year;
3. their maximum life span is three years;
4. rabbits have $0, 6, 8$ baby rabbits in their three years, respectively.

If you know the population one year, what is the population the next year?

$$f_n = \text{first-year rabbits in year } n$$
$$s_n = \text{second-year rabbits in year } n$$
$$t_n = \text{third-year rabbits in year } n$$

The rules say:

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ s_{n+1} \\ t_{n+1} \end{pmatrix}.$$

Let $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$ and $v_n = \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix}$. Then $\boxed{Av_n = v_{n+1}}$. $\longleftarrow$ difference equation

A Biology Question
Continued

If you know v_0, what is v_{10}?

$$v_{10} = Av_9 = AAv_8 = \cdots = A^{10}v_0.$$

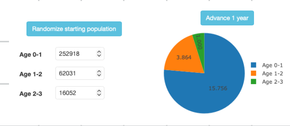
This makes it easy to compute examples by computer: [interactive]

v_0	v_{10}	v_{11}
$\begin{pmatrix} 3 \\ 7 \\ 9 \end{pmatrix}$	$\begin{pmatrix} 30189 \\ 7761 \\ 1844 \end{pmatrix}$	$\begin{pmatrix} 61316 \\ 15095 \\ 3881 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 9459 \\ 2434 \\ 577 \end{pmatrix}$	$\begin{pmatrix} 19222 \\ 4729 \\ 1217 \end{pmatrix}$
$\begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix}$	$\begin{pmatrix} 28856 \\ 7405 \\ 1765 \end{pmatrix}$	$\begin{pmatrix} 58550 \\ 14428 \\ 3703 \end{pmatrix}$

What do you notice about these numbers?

1. Eventually, each segment of the population doubles every year: $Av_n = v_{n+1} = 2v_n$.
2. The ratios get close to $(16 : 4 : 1)$:

$$v_n = (\text{scalar}) \cdot \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}.$$



Problem: One of the eigenvalues of A is $\lambda = 8$. Identify the other eigenvalue and find at least two eigenvectors for each eigenvalue for the matrix $A = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$

Important

If x is a nonzero vector in Nul(A), then x is an eigenvector of A with eigenvalue $\lambda = 0$.

Consequently, non-invertible matrices always have $\lambda = 0$ as an eigenvalue.

Eigenspaces

Definition

Let A be an $n\times n$ matrix and let λ be an eigenvalue of A. The λ-**eigenspace** of A is the set of all eigenvectors of A with eigenvalue λ, plus the zero vector:

$$\begin{aligned}\lambda\text{-eigenspace} &= \{v \text{ in } \mathbf{R}^n \mid Av = \lambda v\}\\ &= \{v \text{ in } \mathbf{R}^n \mid (A-\lambda I)v = 0\}\\ &= \mathrm{Nul}(A-\lambda I).\end{aligned}$$

Since the λ-eigenspace is a null space, it is a *subspace* of $\mathbf{R}^n$.

How do you find a basis for the λ-eigenspace? Parametric vector form!

Eigenspaces

Example

Find a basis for the $\underset{\lambda}{3}$-eigenspace of

$$A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}.$$

Eigenspaces

Example

Find a basis for the $\underset{\lambda}{2}$-eigenspace of

$$A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}.$$

Question: how many eigenvalues can a matrix have? Can they have more than one eigenvalue?

Eigenspaces
Example

Find a basis for the $\frac{1}{2}$-eigenspace of

$$A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}.$$

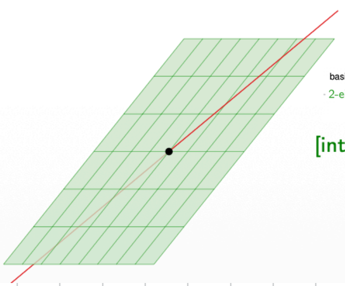
https://textbooks.math.gatech.edu/ila/demos/eigenspace.html?nomult

Eigenspaces
Example: picture

$$A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}.$$

We computed bases for the 2-eigenspace and the 1/2-eigenspace:

basis for the $\frac{1}{2}$-eigenspace: $\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$



basis for the 2-eigenspace: $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$

[interactive]

Eigenspaces

Geometry; example

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!

https://textbooks.math.gatech.edu/ila/demos/eigenspace.html?mat=0,-1:-1,0&nomult

Eigenspaces

Geometry; example

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the vertical projection onto the x-axis, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!

https://textbooks.math.gatech.edu/ila/demos/eigenspace.html?mat=1,0:0,0&nomult

A Matrix is Invertible if and only if Zero is not an Eigenvalue

Fact: A is invertible if and only if 0 is not an eigenvalue of A.

Why?

$$\begin{aligned} 0 \text{ is an eigenvalue of } A &\iff Ax = 0x \text{ has a nontrivial solution} \\ &\iff Ax = 0 \text{ has a nontrivial solution} \\ &\iff A \text{ is not invertible.} \end{aligned}$$

invertible matrix theorem

The Invertible Matrix Theorem

Addenda

We have a couple of new ways of saying "A is invertible" now:

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix, and let $T: \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. A is invertible.
2. T is invertible.
3. The reduced row echelon form of A is I_n.
4. A has n pivots.
5. $Ax = 0$ has no solutions other than the trivial one.
6. $\text{Nul}(A) = \{0\}$.
7. $\text{nullity}(A) = 0$.
8. The columns of A are linearly independent.
9. The columns of A form a basis for $\mathbf{R}^n$.
10. T is one-to-one.
11. $Ax = b$ is consistent for all b in $\mathbf{R}^n$.
12. $Ax = b$ has a unique solution for each b in $\mathbf{R}^n$.
13. The columns of A span $\mathbf{R}^n$.
14. $\text{Col}\, A = \mathbf{R}^m$.
15. $\dim \text{Col}\, A = m$.
16. $\text{rank}\, A = m$.
17. T is onto.
18. There exists a matrix B such that $AB = I_n$.
19. There exists a matrix B such that $BA = I_n$.
20. The determinant of A is *not* equal to zero.
21. The number 0 is *not* an eigenvalue of A.

Eigenvectors with Distinct Eigenvalues are Linearly Independent

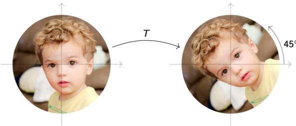
Fact: If $v_1, v_2, \ldots, v_k$ are eigenvectors of A with *distinct* eigenvalues $\lambda_1, \ldots, \lambda_k$, then $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

Warning! Knowing vectors v1 and v2 have the **same** eigenvalue does NOT mean that they are linearly dependent!

You Try It!

Poll

Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be counterclockwise rotation by $45°$, and let A be the matrix for T.



Poll

Find an eigenvector of A without doing any computations.

A. Okay. **B.** No way.

Eigenspaces

Summary

Let A be an $n \times n$ matrix and let λ be a number.

1. λ is an eigenvalue of A if and only if $(A - \lambda I)x = 0$ has a nontrivial solution, if and only if $\text{Nul}(A - \lambda I) \neq \{0\}$.
2. In this case, finding a basis for the λ-eigenspace of A means finding a basis for $\text{Nul}(A - \lambda I)$ as usual, i.e. by finding the parametric vector form for the general solution to $(A - \lambda I)x = 0$.
3. The eigenvectors with eigenvalue λ are the nonzero elements of $\text{Nul}(A - \lambda I)$, i.e. the nontrivial solutions to $(A - \lambda I)x = 0$.

Summary

- **Eigenvectors** and **eigenvalues** are the most important concepts in this course.
- Eigenvectors are by definition nonzero; eigenvalues may be zero.
- The eigenvalues of a triangular matrix are the diagonal entries.
- A matrix is invertible if and only if zero is not an eigenvalue.
- Eigenvectors with distinct eigenvalues are linearly independent.
- The λ-eigenspace is the set of all λ-eigenvectors, plus the zero vector.
- You can compute a basis for the λ-eigenspace by finding the parametric vector form of the solutions of $(A - \lambda I_n)x = 0$.

Section 5.2

The Characteristic Polynomial

The Characteristic Polynomial

Definition

Let A be a square matrix. The **characteristic polynomial** of A is

$$f(\lambda) = \det(A - \lambda I).$$

The **characteristic equation** of A is the equation

$$f(\lambda) = \det(A - \lambda I) = 0.$$

Important

The eigenvalues of A are the roots of the characteristic polynomial $f(\lambda) = \det(A - \lambda I)$.

The Characteristic Polynomial

Example

Question: What are the eigenvalues of

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}?$$

You Try It!

Q: What are the eigenvalues of A?

$$A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$$

Problem: Find the characteristic polynomial of A and then find the eigenvalues of the matrix where $A = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix}$

Recall from before

If A is upper or lower triangular, then the determinant of A is the product of the diagonal entries.

Consequently, if A is upper or lower triangular, then the eigenvalues of A are just the diagonal entries.

Warning! This trick does **not work** to find the eigenvalues unless A is upper/lower triangular!

The Characteristic Polynomial

Example

Question: What is the characteristic polynomial of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}?$$

Definition

The **trace** of a square matrix A is $\mathrm{Tr}(A) =$ sum of the diagonal entries of A.

Shortcut

The characteristic polynomial of a 2×2 matrix A is
$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\,\lambda + \det(A)$.

Summary

We did two different things today.

First we talked about the geometry of eigenvalues and eigenvectors:

- Eigenvectors are vectors v such that v and Av are on the same line through the origin.
- You can pick out the eigenvectors geometrically if you have a picture of the associated transformation.

Then we talked about characteristic polynomials:

- We learned to find the eigenvalues of a matrix by computing the roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.
- For a 2×2 matrix A, the characteristic polynomial is just

$$p(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A).$$

Section 5.4

Diagonalization

Motivation
Difference equations

Many real-word linear algebra problems have the form:

$$v_1 = Av_0, \quad v_2 = Av_1 = A^2 v_0, \quad v_3 = Av_2 = A^3 v_0, \quad \dots \quad v_n = Av_{n-1} = A^n v_0.$$

This is called a **difference equation**.

Our toy example about rabbit populations had this form.

The question is, what happens to v_n as $n \to \infty$?

- Taking powers of diagonal matrices is easy!
- Taking powers of *diagonalizable* matrices is still easy!
- Diagonalizing a matrix is an eigenvalue problem.

Powers of Diagonal Matrices

If D is diagonal, then D^n is also diagonal; its diagonal entries are the nth powers of the diagonal entries of D:

$$D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \quad D^2 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \quad D^3 = \begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}, \quad \dots \quad D^n = \begin{pmatrix} 2^n & 0 \\ 0 & (-1)^n \end{pmatrix}.$$

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad D^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{9} \end{pmatrix}, \quad D^3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{27} \end{pmatrix},$$

$$\dots \quad D^n = \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & \frac{1}{2^n} & 0 \\ 0 & 0 & \frac{1}{3^n} \end{pmatrix}$$

Similar Matrices

Definition

Two $n \times n$ matrices are **similar** if there exists an invertible $n \times n$ matrix C such that $A = CBC^{-1}$.

Fact: if two matrices are similar then so are their powers:

$$A = CBC^{-1} \quad \Longrightarrow \quad A^n = CB^nC^{-1}.$$

for n=3 and n=6

Powers of Matrices that are Similar to Diagonal Ones

What if A is not diagonal?

Example

Let $A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}$. Compute A^n, using

$$A = CDC^{-1} \quad \text{for} \quad C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

(continued on next page)

for n=3 and n=6

Powers of Matrices that are Similar to Diagonal Ones

What if A is not diagonal?

Example

Let $A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}$. Compute A^n, using

$$A = CDC^{-1} \quad \text{for} \quad C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

Diagonalizable Matrices

Definition

An $n \times n$ matrix A is **diagonalizable** if it is similar to a diagonal matrix:

$$A = CDC^{-1} \quad \text{for } D \text{ diagonal.}$$

Important

If $A = CDC^{-1}$ for $D = \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix}$ then

$$A^k = CD^kC^{-1} = C \begin{pmatrix} d_{11}^k & 0 & 0 \\ 0 & d_{22}^k & 0 \\ 0 & 0 & d_{33}^k \end{pmatrix} C^{-1}.$$

Diagonalization

The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors in $\mathbf{R}^n$.

In this case, $A = CDC^{-1}$ for

$$C = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where $v_1, v_2, \ldots, v_n$ are linearly independent eigenvectors, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the corresponding eigenvalues (in the same order).

Corollary ← a theorem that follows easily from another theorem

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Diagonalization

Procedure

How to diagonalize a matrix A:

1. Find the eigenvalues of A using the characteristic polynomial.
2. For each eigenvalue λ of A, compute a basis $\mathcal{B}_\lambda$ for the λ-eigenspace.
3. If there are fewer than n total vectors in the union of all of the eigenspace bases $\mathcal{B}_\lambda$, then the matrix is not diagonalizable.
4. Otherwise, the n vectors $v_1, v_2, \ldots, v_n$ in your eigenspace bases are linearly independent, and $A = CDC^{-1}$ for

$$C = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where λ_i is the eigenvalue for v_i.

Warning! Knowing A has **repeated** eigenvalues does NOT mean that it is not diagonalizable!

Diagonalization
Example

Problem: Diagonalize $A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}$.

The characteristic polynomial is

$$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\,\lambda + \det(A) = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2).$$

Diagonalization
Another example

Problem: Diagonalize $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

The characteristic polynomial is

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -(\lambda-1)^2(\lambda-2).$$

Algebraic Multiplicity

Definition

The **(algebraic) multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

Diagonalization
Procedure

The Diagonalization Theorem (Alternate Form)

Let A be an $n \times n$ matrix. The following are equivalent:

1. A is diagonalizable.
2. The sum of the geometric multiplicities of the eigenvalues of A equals n.
3. The sum of the algebraic multiplicities of the eigenvalues of A equals n, and for each eigenvalue, *the geometric multiplicity equals the algebraic multiplicity*.

Non-Distinct Eigenvalues
Examples

Example

If A has n distinct eigenvalues, then the algebraic multiplicity of each equals 1, hence so does the geometric multiplicity, and therefore A is diagonalizable.

For example, $A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}$ has eigenvalues -1 and 2, so it is diagonalizable.

Example

The matrix $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$ has characteristic polynomial

$$f(\lambda) = -(\lambda - 1)^2(\lambda - 2).$$

The algebraic multiplicities of 1 and 2 are 2 and 1, respectively. They sum to 3. We showed before that the geometric multiplicity of 1 is 2 (the 1-eigenspace has dimension 2). The eigenvalue 2 automatically has geometric multiplicity 1. Hence the geometric multiplicities add up to 3, so A is diagonalizable.

Diagonalization
A non-diagonalizable matrix

Problem: Show that $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

You Try It!

Poll

Poll

Which of the following matrices are diagonalizable, and why?

A. $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ B. $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ C. $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ D. $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Algebraic Multiplicity

Definition

The **(algebraic) multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

Non-Distinct Eigenvalues

Definition

Let λ be an eigenvalue of a square matrix A. The **geometric multiplicity** of λ is the dimension of the λ-eigenspace.

Theorem

Let λ be an eigenvalue of a square matrix A. Then

$$1 \leq (\text{the geometric multiplicity of } \lambda) \leq (\text{the algebraic multiplicity of } \lambda).$$

More examples!

Problem: Diagonalize the matrix $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Problem: Diagonalize the matrix $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$ which has characteristic polynomial $p(\lambda) = -\lambda^3 - 3\lambda^2 + 4 = -(\lambda - 1)(\lambda + 2)^2$

$$A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$$

You Try It!

Poll

Poll

Which of the following matrices are diagonalizable, and why?

A. $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ B. $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ C. $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ D. $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Summary

- A matrix A is **diagonalizable** if it is similar to a diagonal matrix D: $A = CDC^{-1}$.
- It is easy to take powers of diagonalizable matrices: $A^r = CD^rC^{-1}$.
- An $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors $v_1, v_2, \ldots, v_n$ in $\mathbf{R}^n$, in which case $A = CDC^{-1}$ for

$$C = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

- If A has n distinct real eigenvalues, then it is diagonalizable.
- The **geometric multiplicity** of an eigenvalue λ is the dimension of the λ-eigenspace.
- $1 \leq$ (geometric multiplicity) $\leq$ (algebraic multiplicity).
- An $n \times n$ matrix is diagonalizable if and only if the sum of the geometric multiplicities is n.