Worksheet 10

1. Consider the vectors

$$\vec{u} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \ \vec{v} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

   (a) Are $\vec{u}$ and $\vec{v}$ orthogonal? **Solution**: They are orthogonal only if $\vec{u} \cdot \vec{v} = 0$. Here, $\vec{u} \cdot \vec{v} = (1)(2) + (-3)(2) + (2)(1) = -2$, so they are not orthogonal.

   (b) Find the orthogonal projection of $\vec{v}$ onto $\vec{u}$, then find the component of $\vec{v}$ perpendicular to $\vec{u}$. **Solution**: The orthogonal projection of $\vec{v}$ onto $\vec{u}$ is given by

$$\hat{v} = \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u} = \frac{-2}{14} \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/7 \\ 3/7 \\ -2/7 \end{bmatrix}.$$

   The component of $\vec{v}$ perpendicular to $\vec{u}$ is

$$\vec{z} = \vec{v} - \hat{v} = \begin{bmatrix} 15/7 \\ 11/7 \\ 9/7 \end{bmatrix}$$

   (c) Let the component of $\vec{v}$ perpendicular to $\vec{u}$ be called $\vec{z}$. Is the matrix $M = [\vec{u}\ \vec{v}\ \vec{z}]$ invertible? Explain. **Solution**: By definition of what $\vec{z}$ is, we know that $\vec{v} = \alpha\vec{u} + \vec{z}$, where $\alpha\vec{u}$ is the orthogonal projection of $\vec{v}$ onto $\vec{u}$. But, that means that $\vec{v}$ is a linear combination of the vectors $\vec{u}$ and $\vec{z}$, so the columns of M are linearly dependent, so M is not invertible.

   (d) For the matrix M in part (c), find a nontrivial solution to $M\vec{x} = \vec{0}$. **Solution**: The weights of any nontrivial linear combination of $\vec{u}$, $\vec{v}$, and $\vec{z}$ that equals $\vec{0}$ will work. As stated above, we know that $\alpha\vec{u} - \vec{v} + \vec{z} = \vec{0}$, so one potential answer is the vector $\vec{x} = (-1/7,\ -1,\ 1)$. There are infinite answers, though, since M is not invertible.

2. Answer the following short questions, being sure to explain answers fully:

(a) Is every linearly independent set of nonzero vectors in $\mathbb{R}^n$ orthogonal? **Solution**: No, though the opposite is true - every orthogonal set of nonzero vectors is linearly independent. As a simple counterexample to the question asked, consider vectors (1, 1) and (1, 0). These two nonzero vectors are linearly independent, as one is not a constant multiple of the other, but they are not orthogonal, as the dot product of the two is $(1)(1)+(1)(0) = 1 \neq 0$.

(b) Describe the subspace spanned by a set of n orthogonal nonzero vectors in $\mathbb{R}^n$. **Solution**: Since the set is orthogonal, it is also linearly independent. Any set of n linearly independent vectors in $\mathbb{R}^n$ spans all of $\mathbb{R}^n$, so the span is simply $\mathbb{R}^n$.

(c) Matrix M has eigenvalue 0, and its columns form an orthogonal set. Give a vector that must be one of the columns of M. **Solution**: Since M has eigenvalue 0, it is not invertible, and its columns are linearly dependent. However, the columns are orthogonal. All orthogonal sets of nonzero vectors are linearly independent, so that means that the set of vectors that is the columns of M must not consist of all nonzero vectors. At the same time, the zero vector is orthogonal to all other vectors. Hence, M must have the zero vector as one of its columns.

(d) Take two nonzero vectors $\vec{u}$ and $\vec{y}$ in $\mathbb{R}^n$. If the projection of $\vec{u}$ onto the subspace spanned by $\vec{y}$ is the same as the projection of $\vec{y}$ onto the subspace spanned by $\vec{u}$, how are $\vec{u}$ and $\vec{y}$ related geometrically? **Solution**: The projections of $\vec{u}$ onto the subspace spanned by $\vec{y}$ and $\vec{y}$ onto the subspace spanned by $\vec{u}$ are

$$\hat{u} = \frac{\vec{y} \cdot \vec{u}}{||\vec{y}||^2}\vec{y}, \ \hat{y} = \frac{\vec{y} \cdot \vec{u}}{||\vec{u}||^2}\vec{u} \ .$$

For these two to be the same ($\hat{u} = \hat{y}$), you would either need $\vec{u} \cdot \vec{y} = 0$ or $\vec{y}/||\vec{y}||^2 = \vec{u}/||\vec{u}||^2$. The former would mean that $\vec{y}$ and $\vec{u}$ are perpendicular, and the latter would mean that $\vec{y}$ and $\vec{u}$ are parallel (and are equal, in fact).