Worksheet 11 (Last one!)

1. Let $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $W = \text{span}\{\vec{u}_1,\ \vec{u}_2\}$. Find the projection of $\vec{y} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ onto $W^{\perp}$, then use your answer to find an orthonormal basis for $W^{\perp}$.

Solution: Finding the projection of $\vec{y}$ onto $W^{\perp}$, is the same as finding the component of $\vec{y}$ orthogonal to W, which is $\vec{y} - \text{proj}_W\vec{y}$. So, we need to find $\text{proj}_W\vec{y}$, which will require an orthogonal basis for W (or see answer to problem 2). Our current basis vectors $\vec{u}_1$ and $\vec{u}_2$ are (as they must be) linearly independent, but not orthogonal, since $\vec{u}_1 \cdot \vec{u}_2 = 1 \neq 0$. We can find an orthogonal basis using Gram-Schmidt, though. Let the first orthogonal basis vector be $\vec{v}_1 = \vec{u}_1$. Then the second orthogonal basis vector will be

$$\vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 2 \end{bmatrix}.$$

For the sake of removing fractions, we will actually let $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ (just multiplying the result above by the scalar 2), though this is not necessary of course. Then, to find $\text{proj}_W\vec{y}$, we do

$$\text{proj}_W\vec{y} = \frac{\vec{y} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 + \frac{\vec{y} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{15}{18}\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -1/3 \\ 10/3 \end{bmatrix}.$$

Hence, the projection of $\vec{y}$ onto $W^{\perp}$ is

$$\text{proj}_{W^{\perp}}\vec{y} = \vec{y} - \text{proj}_W\vec{y} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 4/3 \\ -1/3 \\ 10/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3 \\ -1/3 \end{bmatrix}$$

As to finding an orthonormal basis for $W^{\perp}$: note that W is a plane (2 dimensional subspace) in $\mathbb{R}^3$, so $W^{\perp}$ is just a line (1 dimensional subspace) in $\mathbb{R}^3$. So, we only need a single unit vector that is in $W^{\perp}$

in order to have an orthonormal basis for $W^\perp$. We already have such a vector: $\text{proj}_{W^\perp}\vec{y}$ (note that $||\text{proj}_{W^\perp}\vec{y}|| = 1$, so it is already a unit vector).

2. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \ \vec{z} = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix} .$$

Find the distance between col$A$ and $\vec{z}$ without finding an orthogonal basis for col$A$.

Solution: The easiest way to answer this is probably to note that the question is equivalent to asking what the least squares error is for the problem $A\vec{x} = \vec{z}$, since the closest point to $\vec{z}$ in col$A$ is $\text{proj}_{\text{col}A}\vec{z} = \hat{z}$, the distance between $\vec{z}$ and col$A$ is the length of the vector $\vec{z} - \hat{z}$, and the least squares solution $\hat{x}$ satisfies $A\hat{x} = \hat{z}$. So, we will begin by solving the normal equations for the least squares solution $\hat{x}$, which are $A^T A\hat{x} = A^T\vec{z}$. We find

$$A^T A = \begin{bmatrix} 4 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 3 \end{bmatrix}, \ A^T\vec{z} = \begin{bmatrix} 14 \\ 4 \\ 13 \end{bmatrix} .$$

So, to solve the normal equations, we construct the augmented matrix for the system and row reduce. This gives

$$\begin{bmatrix} 4 & 2 & 3 & 14 \\ 2 & 2 & 1 & 4 \\ 3 & 1 & 3 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} .$$

Hence, our least squares solution is $\hat{x} = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$. To find $\hat{z}$ we simply multiply $A\hat{x} = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 5 \end{bmatrix}$, this tells us that $\vec{z} - \hat{z} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ -3 \end{bmatrix}$. Then the distance between $\vec{z}$ and col$A$ is

$$||\vec{z} - \hat{z}|| = \sqrt{18} = 3\sqrt{2} .$$

Note that we could have solved problem 1 in a similar way through the least squares solution, rather than finding an orthogonal basis for W.