Worksheet 2

1. Consider the following vectors:

$$\vec{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \ \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \ \vec{u}_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}.$$

Is $\vec{b}$ in the span of $\{\vec{u}_1,\ \vec{u}_2,\ \vec{u}_3\}$? Is $\vec{b}$ a linear combination of vectors $\vec{u}_1,\ \vec{u}_2$, and $\vec{u}_3$? Is the linear system with augmented matrix $A = \begin{bmatrix}\vec{u}_1 \ \vec{u}_2 \ \vec{u}_3 \ \vec{b}\end{bmatrix}$ consistent?

Solution: First, note that the answers to all three questions are logically equivalent - if one is true, they are all true, and if any is false, they are all false. So, focus on question three. After constructing A, we can row reduce by replacing row 2 by the sum of itself and 2 times row 1, then we can replace row 3 with the sum of itself minus 2 times row 2. This leaves

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which is in reduced echelon form. Note that the system is consistent, since the rightmost column has no pivot, and that x_1 and x_2 are basic variables while x_3 is a free variable. So, the answer to the third question is "yes", and therefore so are the answers to the other two questions. As a specific example for the second question, note that we could write $\vec{b} = 2\vec{u}_1 + 3\vec{u}_2$, but that there are actually infinite ways to write $\vec{b}$ in terms of our three vectors here, since x_3 is free.

2. Mark each statement as true or false, and justify your answers:

(a) A vector $\vec{b}$ is a linear combination of the columns of a matrix A if and only if the equation $A\vec{x} = \vec{b}$ has at least one solution. **Solution**: True. Since the meaning of $A\vec{x}$ is a linear combination of the columns of A with weights given by the entries of $\vec{x}$, if $\vec{b}$ is such a linear combination, then there is at least one $\vec{x}$ for which $A\vec{x} = \vec{b}$ is true (at least one solution).

(b) The equation $A\vec{x} = \vec{b}$ is consistent if the augmented matrix $\begin{bmatrix}A \ \vec{b}\end{bmatrix}$ has a pivot position in every row. **Solution**: False. While this statement *might* be true for a given A and $\vec{b}$, it is not necessarily true. As a counter-example, if there is a pivot in every row of the augmented matrix, but the pivot in one of the rows (it would have to be the bottom-most row with a pivot) is in the right-most column, then the system is inconsistent. If the question had instead referred to the matrix A directly (and not the augmented matrix) having a pivot in every row, the answer would be true.

(c) The first entry in the product $A\vec{x}$ is a sum of products. **Solution**: True. By definition of what matrix vector multiplication means, the first entry in the product $A\vec{x}$ is $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n$, which is a sum of products.

(d) If the columns of an $m \times n$ matrix A span $\mathbb{R}^m$, then the equation $A\vec{x} = \vec{b}$ is consistent for each $\vec{b}$ in $\mathbb{R}^m$. **Solution**: True. Since the columns of A span $\mathbb{R}^m$, that means that every possible vector $\vec{b}$ in $\mathbb{R}^m$ can be constructed as a linear combination of the columns of A, which in matrix equation form means that $A\vec{x} = \vec{b}$ has at least one solution (is consistent) for all $\vec{b}$ in $\mathbb{R}^m$.

(d) If A is an $m \times n$ matrix and if the equation $A\vec{x} = \vec{b}$ is inconsistent for some $\vec{b}$ in $\mathbb{R}^m$, then A cannot have a pivot position in every row. **Solution**: True. If $A\vec{x} = \vec{b}$ is inconsistent for some $\vec{b}$, that means that the augmented matrix $\begin{bmatrix} A & \vec{b} \end{bmatrix}$ in that case has a pivot in the rightmost column, meaning that all other entries in that particular row are 0 in reduced echelon form. But, if that is true, then for that particular row the matrix A by itself would have all 0s in reduced echelon form, and would therefore not have a pivot in that row. Hence, it could not have a pivot in every row.