Worksheet 3

1. Find the solution of the following system and write it in parametric vector form. Give a geometric description of the solution set.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ -3x_2 - 6x_3 &= -3 \end{aligned}$$

Solution: We first write the augmented matrix for this system. It is

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{bmatrix}.$$

We can row reduce by first replacing row 2 with itself plus 4 times row 1, then replace row 3 with itself plus row 2. At this point we are in echelon form, with

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that there is no pivot in the rightmost column, so the system is consistent, and there is also no pivot in the column corresponding to x_3, so it is a free variable. To put in reduced echelon form, we scale row 2 by dividing by 3, then replace row 1 with itself minus 3 times row 2. This gives

$$\begin{bmatrix} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So, the solutions are $x_1 = -2+5x_3$, $x_2 = 1-2x_3$, x_3 free. In parametric vector form, we would have

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}.$$

Geometrically, the solution is represented by a line in $\mathbf{R}^3$ going through the point $(-2, 1, 0)$ and parallel to the vector $[5 \ \ -2 \ \ 1]$.

2. For the following situations, determine (a) whether the equation $A\vec{x} = \vec{0}$ has a nontrivial solution and (b) whether the equation $A\vec{x} = \vec{b}$ has at least one solution for every possible $\vec{b}$ in $\mathbf{R}^m$, and explain:

(i) A is a 3×3 matrix with 3 pivots. **Solution**: (a) A has a pivot in every column, so there are no nontrivial solutions since there are no free variables. (b) A has a pivot in every row, so there is at least one solution for every $\vec{b}$ in $\mathbf{R}^3$ - in fact, there is exactly one solution for every $\vec{b}$, since there are no free variables.

(ii) A is a 3×3 matrix with 2 pivots. **Solution**: (a) A does not have a pivot in every column, so there are nontrivial solutions since there are free variables. (b) A does not have a pivot in every row, so there is not a solution for each $\vec{b}$ in $\mathbf{R}^3$.

(iii) A is a 3×2 matrix with 2 pivots. **Solution**: (a) A has a pivot in every column, so there are no nontrivial solutions since there are no free variables. (b) A does not have a pivot in every row, so there is not a solution for each $\vec{b}$ in $\mathbf{R}^3$.

(iv) A is a 2×4 matrix with 2 pivots. **Solution**: (a) A does not have a pivot in every column, so there are nontrivial solutions since there are free variables. (b) A has a pivot in every row, so there is at least one solution for every $\vec{b}$ in $\mathbf{R}^3$ - in fact, there are infinite solutions for every $\vec{b}$, since there are free variables.