Worksheet 4

1. Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 4 \\ -7 \\ 9 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$$

Note that the sum of $\vec{v}_1$ and twice $\vec{v}_2$ is equal to $\vec{v}_3$. Denote A as the matrix $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$. Answer the following questions, with justification:

(a) Is the set of vectors $\{\vec{v}_1, \ \vec{v}_2, \ \vec{v}_3\}$ linearly independent? **Solution**: No. Since $\vec{v}_3$ is a linear combination of the vectors $\vec{v}_1$ and $\vec{v}_2$ – $\vec{v}_3 = \vec{v}_1 + 2\vec{v}_2$ specifically – the set is linearly dependent.

(b) Without row reducing, does A have a pivot in every column? **Solution**: No. Since the vectors are linearly dependent, the matrix A with those vectors as its columns must have free variables, and therefore does not have a pivot in every column.

(c) Does the linear transform $T(\vec{x}) = A\vec{x}$ map $\mathbb{R}^3$ onto $\mathbb{R}^3$? Is this linear transform one-to-one? **Solution**: No to both. Since the matrix A does not have a pivot in every column, it also does not have one in every row, as it is a square matrix. Since there is not a pivot in every row, there will be some vectors $\vec{b}$ in $\mathbb{R}^3$ whereby $A\vec{x} = \vec{b}$ will not have a solution; hence, not every vector in the co-domain is part of the range of the transform, so the transform is not "onto". In addition, the lack of a pivot in every column means there are free variables, so that even for those $\vec{b}$ that are in the range of T, there are infinite solutions $\vec{x}$ to achieve those $\vec{b}$, so the transform is not one-to-one.

(d) Again, without row reducing, find a non-trivial solution of $A\vec{x} = \vec{0}$. **Solution**: We already know the linear relationship between the vectors: $\vec{v}_3 = \vec{v}_1 + 2\vec{v}_2$. This can be rewritten in a homogeneous form as $\vec{v}_1 + 2\vec{v}_2 - \vec{v}_3 = \vec{0}$ (or the negative of this). This is a non-trivial linear combination of the vectors that is equal to $\vec{0}$, so the coefficients of the vectors in this equation are the entries of our non-trivial solution $\vec{x}$. Hence, $\vec{x} = [1 \ 2 \ -1]$.

2. Let the linear transform $T : \ \mathbb{R}^2 \rightarrow \mathbb{R}^2$ first reflect points across the horizontal x_1 axis and then reflect them across the line $x_2 = x_1$. Find the standard matrix representation of T.

Solution: In order to find the standard matrix of a linear transform, we need only determine how it acts on the columns of the identity matrix I_n. Here we have $n = 2$, so we need to consider $\vec{e}_1 = [1\ 0]$ and $\vec{e}_2 = [0\ 1]$. Take $\vec{e}_1$. T first reflects this across the x_1 axis, but since the vector is already on that axis, nothing happens. Then, the point is reflected across the diagonal line $x_2 = x_1$, which will move $[1\ 0]$ to point $[0\ 1]$. So, $T(\vec{e}_1) = [0\ 1]$. Now take $\vec{e}_2$. T will first reflect across the x_1 axis, turning $\vec{e}_2$ into $[0\ \ -1]$. Then T will reflect across the $x_2 = x_1$ line, putting the point at $[-1\ 0]$. So, $T(\vec{e}_2) = [-1\ 0]$. The matrix of T is $A = [T(\vec{e}_1)\ T(\vec{e}_2)]$, giving

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$