Worksheet 5

1. Consider the matrices

$$A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix} .$$

For what value(s) of k, if any, do matrices A and B commute?

Solution: The two matrices commute if $AB = BA$. So, compute AB and BA to find

$$AB = \begin{bmatrix} 23 & -10+5k \\ -9 & 15+k \end{bmatrix}, \; BA = \begin{bmatrix} 23 & 15 \\ 6-3k & 15+k \end{bmatrix} .$$

The entries at (1,1) and (2,2) are equal no matter what k is. The entries at (1,2) are equal only if $k = 5$, and the entries at (2,1) are also only equal if $k = 5$. So, the matrices commute if and only if $k = 5$.

2. Suppose P is an invertible matrix, and $A = PBP^{-1}$, where A and B are also matrices. Solve for B in terms of A.

Solution: To solve for B, we want to remove the P and P^{-1} that are sandwiching it. To do this, we can left multiply both sides of the equation by P^{-1} and right multiply both sides by P. This leaves

$$P^{-1}AP = P^{-1}PBP^{-1}P .$$

On the right hand side, the two factors of $P^{-1}P$ are both equal to the identity, so that just gives

$$P^{-1}AP = IBI = B ,$$

and that is the answer.

3. Find the inverse of the following matrix, if it exists:

$$\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

Solution: Create the augmented matrix $[A|I_3]$. Then row reduce the entire matrix so that the left hand side that starts as A is converted into the identity matrix I_3. Here are some steps that do this: 1) replace row 2 by itself plus 3 times row 1, 2) replace row 3 by itself minus 2 times row 1, 3) replace row 3 with itself plus 3 times row 2 (at this point A is in echelon form), 4) replace row 2 with itself plus row 3, 5) replace row 1 with itself plus row 3, and 6) scale row 3 by dividing by 2. This gives

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right];$$

the left half is I_3, so the right half of this matrix is then A^{-1}.

4. Answer the following short questions, justifying your answers. Note that all referenced matrices are $n \times n$ square.

   (a) Matrix A has one column that is 7 times another column. Is A invertible? **Solution**: If one of the columns of A is 7 times another column, then the columns are not linearly independent, so there is not a pivot in every column, so A is not invertible.

   (b) A is the standard matrix of a linear transform $T : \mathbb{R}^n \to \mathbb{R}^n$ that is one-to-one. Is A invertible? **Solution**: Since the transform T is one-to-one, there are no free variables in A, so there is a pivot in every column, so A is invertible.

   (c) A is not invertible. How many solutions are there to the equation $A\vec{x} = \vec{0}$? **Solution**: Since A is not invertible, it does not have a pivot in every column, so there are free variables, and there are infinite solutions to $A\vec{x} = \vec{0}$.

   (d) There are some vectors in $\mathbb{R}^n$ that are not in the span of the columns of A. Is A invertible? **Solution**: Since there are vectors that are not in the span of the columns of A, there are some vectors for which $A\vec{x} = \vec{b}$ is inconsistent (has no solution), so there is not a pivot in every row of A, so there is not a pivot in every column, so A is not invertible.