Worksheet 8

1. Find a basis for the eigenspace of

$$A = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}$$

with eigenvalue 10.

Solution: A basis for the eigenspace would be a linearly independent set of vectors that solve $(A - 10I_2)\vec{v} = \vec{0}$; that is, null space basis vectors for matrix $(A - 10I_2)$. To find what these are, we first compute

$$(A - 10I_2) = \begin{bmatrix} -6 & -2 \\ -3 & -1 \end{bmatrix} ;$$

note that the columns of this matrix are linearly dependent, as must be the case for the null space of this matrix (corresponding the the eigenspace of A) to contain more than just the vector $\vec{0}$. We row reduce the above matrix by replacing row 2 with itself minus one half of row 1, then scaling row 1 by dividing by -6 to get

$$(A - 10I_2) \sim \begin{bmatrix} 1 & 1/3 \\ 0 & 0 \end{bmatrix} .$$

So, component v_2 is free, while $v_1 = -v_2/3$. In parametric vector form, then,

$$\vec{v} = v_2 \begin{bmatrix} -1/3 \\ 1 \end{bmatrix} .$$

Hence, a potential basis vector for the eigenspace of A with eigenvalue 10 is $(-1/3,\ 1)$, but any nonzero multiple of this vector could also be a basis vector.

2. Let $\vec{u}$ and $\vec{v}$ both be eigenvectors of 2×2 matrix A with real eigenvalues λ and μ, respectively, and $\lambda \neq \mu$.

   (a) Explain why the set of vectors $e = \{\vec{u}, \vec{v}\}$ can serve as a basis for $\mathbb{R}^2$. **Solution**: Since the eigenvalues of these eigenvectors are different, $\vec{u}$ and $\vec{v}$ must be linearly independent. Further, set e contains two vectors, both of which are in $\mathbb{R}^2$ (since A is 2×2). Since any set of n linearly independent vectors in $\mathbb{R}^n$ can serve as a basis for $\mathbb{R}^n$, e can serve as a basis for $\mathbb{R}^2$.

(b) If the coordinates of a vector $\vec{x}$ in $\mathbb{R}^2$ relative to the basis e are $(c_1,\ c_2)$, what are the coordinates of the vector $A\vec{x}$ relative to basis e? **Solution**: Given the coordinates of $\vec{x}$ relative to basis e, we know that $\vec{x} = c_1\vec{u} + c_2\vec{v}$. Hence, the vector $A\vec{x}$ can be found by applying A to this equation: $A\vec{x} = c_1 A\vec{u} + c_2 A\vec{v}$. But, since $\vec{u}$ and $\vec{v}$ are eigenvectors of A, the terms $A\vec{u}$ and $A\vec{v}$ give $\lambda\vec{u}$ and $\mu\vec{v}$, respectively. Hence, $A\vec{x} = c_1\lambda\vec{u} + c_2\mu\vec{v}$, which is an equation expressing vector $A\vec{x}$ as a linear combination of the basis vectors of e, with weights $c_1\lambda$ and $c_2\mu$. So, by definition, the coordinates of $A\vec{x}$ relative to the basis e are $(c_1\lambda,\ c_2\mu)$.

(c) If $\lambda = 0$, what is the rank of A? **Solution**: If $\lambda = 0$, then A has an eigenvalue of 0, which means A is not invertible. Hence, its columns are linearly dependent, so it has at most 1 pivot column (since A has only 2 columns total), so its rank is at most 1. A could potentially have zero pivot columns, but only if it were the zero matrix (all zero entries). But, since $\mu \neq \lambda$, then $\mu \neq 0$, so $A\vec{v} \neq \vec{0}$, so A cannot be the zero matrix, so A does not have zero pivot columns. Hence, A has 1 pivot column, and is rank 1.

3. Consider the matrix

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Find the characteristic equation for the eigenvalues of A, and then solve this equation, giving the eigenvalues and their multiplicities.

Solution: The characteristic equation is $\det(A - \lambda I_3) = 0$. In this case, we have

$$A = \det\left(\begin{bmatrix} 3-\lambda & 4 & 5 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{bmatrix}\right) = 0.$$

The determinant is easy to do using a cofactor expansion down the first column, and is just $(3-\lambda)\left[(2-\lambda)^2 - 1\right]$. Expanding out the term in [ ] gives $\lambda^2 - 4\lambda + 3$, which can be factored to $(\lambda - 3)(\lambda - 1)$. Hence, the characteristic equation is $(3-\lambda)(\lambda-3)(\lambda-1) = 0$. So, the solutions are $\lambda = 1$ with multiplicity 1 and $\lambda = 3$ with multiplicity 2.