Worksheet 9

1. Diagonalize the matrix

$$A = \begin{bmatrix} 6 & 3 \\ -4 & -1 \end{bmatrix}$$

if possible, or explain why A cannot be diagonalized.

Solution: First we find eigenvalues, by satisfying the characteristic equation

$$\det\left(\begin{bmatrix} 6-\lambda & 3 \\ -4 & -1-\lambda \end{bmatrix}\right) = 0 \ .$$

Expanding the determinant gives $(6-\lambda)(-1-\lambda)+12 = \lambda^2 - 5\lambda + 6 = (\lambda-3)(\lambda-2)$. Hence, the roots of the characteristic polynomial are at $\lambda = 3,\ 2$. Since there are two distinct eigenvalues here for our 2×2 matrix, we know right away that it is diagonalizable, as the eigenvectors are guaranteed to be linearly independent. To find eigenvectors, we start with $\lambda = 3$ and solve

$$\begin{bmatrix} 6-3 & 3 \\ -4 & -1-3 \end{bmatrix} \vec{v} = \vec{0} \ .$$

This matrix row reduces easily to

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \ ,$$

so v_2 is free and $v_1 = -v_2$; hence, an eigenvector for eigenvalue 3 is $\vec{v} = (-1, 1)$, or any multiple thereof. For the second eigenvector we solve

$$\begin{bmatrix} 6-2 & 3 \\ -4 & -1-2 \end{bmatrix} \vec{v} = \vec{0} \ .$$

This matrix row reduces easily to

$$\begin{bmatrix} 1 & 3/4 \\ 0 & 0 \end{bmatrix} \ ,$$

so v_2 is free and $v_1 = -3/4 v_2$; hence, an eigenvector for eigenvalue 2 is $\vec{v} = (-3/4, 1)$, or any multiple thereof. We will in fact use a multiple here to remove any fractions, and choose $\vec{v} = (-3, 4)$. Thus, matrix A has been diagonalized via

$$P = \begin{bmatrix} -1 & -3 \\ 1 & 4 \end{bmatrix} \ , \ D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

2. Answer the following short questions, being sure to explain answers fully:

   (a) Can a 3×3 matrix have all complex eigenvalues?
   Solution: No, since complex eigenvalues always come in pairs (complex conjugates), there will always be an even number of complex eigenvalues, if any. A 3×3 matrix has three eigenvalues – therefore, so they could not all be complex, as that would indicate an odd number of complex eigenvalues.

   (b) If 2×2 matrix A is not invertible, can it have complex eigenvalues?
   Solution: No. Because A is not invertible, it has 0 as an eigenvalue, leaving only a single other eigenvalue undetermined. Since complex eigenvalues always show up in pairs, that single remaining eigenvalue cannot be complex.

   (c) If A is a real matrix with complex eigenvalue $\lambda = a+ib$ (where $b \neq 0$), can the eigenvector $\vec{v}$ associated with λ be purely imaginary; that is, can $\vec{v} = i\vec{x}$, with $\vec{x}$ in $\mathbb{R}^n$?
   Solution: No. If λ is an eigenvalue of A with eigenvector $\vec{v}$, then $A\vec{v} = \lambda\vec{v}$. If A is real but $\vec{v}$ is purely imaginary, then $A\vec{v}$ is also purely imaginary. But in this case, $\lambda\vec{v} = (a+ib)i\vec{x} = -b\vec{x} + ia\vec{x}$, which definitely has a real part, namely $-b\vec{x}$, since $b \neq 0$ and $\vec{x} \neq \vec{0}$. So, if $\vec{v}$ were purely imaginary, $A\vec{v}$ could not possibly equal $\lambda\vec{v}$ (given a complex λ), so $\vec{v}$ could not be an eigenvector of A with eigenvalue λ.

   (d) If 2×2 real matrix A has purely imaginary eigenvalues ib and $-ib$ (and assume $b > 0$), by what angle does the similar rotation/scaling matrix $C = P^{-1}AP$ discussed in class rotate vectors, and by what factor does it scale them? Be sure to answer for both of the eigenvalues.
   Solution: The similar rotation/scaling matrix for the general case of complex eigenvalue $a - ib$ is

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

   So for our eigenvalue $-ib$,

$$C = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} = b \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

So, the scaling is by the factor b (this is the case for both eigenvalues, as we can always factor out b from both). The rotation matrix is

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

which represents rotation by $\pi/2$ counterclockwise; for the eigenvalue ib, this would be rotation by $\pi/2$ clockwise, as all the signs of our rotation matrix would flip.