Worksheet 6

1. Consider the matrix
$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 5 & -9 \\ 4 & 1 & 2 \end{bmatrix}.$$
Find bases for the column space of A and the null space of A. What are the dimensions of col$A$ and null$A$? Describe col$A$ and null$A$ geometrically. What is the rank of A?

2. Consider the vectors
$$\vec{b}_1 = \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}, \ \vec{b}_2 = \begin{bmatrix} -3 \\ -7 \\ 5 \end{bmatrix}, \ \vec{x} = \begin{bmatrix} 4 \\ 10 \\ -7 \end{bmatrix}.$$
Explain why the set $\beta = \{\vec{b}_1, \ \vec{b}_2\}$ can be considered as a basis for a subspace H. Geometrically describe H. Is $\vec{x}$ in H? If so, give the coordinates of $\vec{x}$ relative to the basis β.

3. Answer the following short questions, justifying your answers fully:
   (a) If M is a 3×5 matrix, and its column space is $\mathbb{R}^3$, does that mean its null space is $\mathbb{R}^2$? If so, explain why, if not, explain what the null space of M actually is.
   (b) Suppose β is a set of vectors that is a basis for a subspace H. If I create a new set of vectors α that includes all of the vectors in β, but also includes one more vector that is a linear combination of some of the vectors of β, is the span of the set α equal to H?
   (c) Is it possible for the null space of an $m \times n$ matrix to be $\mathbb{R}^n$? If so, under what circumstances?