Worksheet 6

1. Consider the matrix

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 5 & -9 \\ 4 & 1 & 2 \end{bmatrix} .$$

Find bases for the column space of A and the null space of A. What are the dimensions of col$A$ and null$A$? Describe col$A$ and null$A$ geometrically. What is the rank of A?

Solution: We can answer this by first row reducing A to reduced echelon form. To do this, interchange row 2 with row 1, then replace row 2 with itself minus 3 times row 1, then replace row three with itself minus 4 times row 1, then scale row 2 by dividing by -13, then scale row 3 by dividing by -19, then replace row 3 with itself minus row 2 (you are now in echelon form), then finally replace row 1 with itself minus 5 times row 2 (now in reduced echelon form), leaving

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} .$$

We see now that the pivot columns of A are columns 1 and 2, so those two columns serve as a basis for col$A$; that is, col$A$=span$\{(3, 1, 4), (2, 5, 1)\}$. For the null space, note that the parametric vector form solution to $A\vec{x} = \vec{0}$, from our reduced echelon form for A, is

$$\vec{x} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} .$$

So, the single vector (-1,2,1) serves as a basis for null$A$.

Since there are two basis vectors for col$A$, its dimension is 2; it is a plane that is a subspace of $\mathbb{R}^3$. Since there is only a single basis vector in null$A$, its dimension is 1; it is a line that is a subspace of $\mathbb{R}^3$. The rank of a matrix is the dimension of its column space, so A is rank 2.

2. Consider the vectors

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix} , \vec{b}_2 = \begin{bmatrix} -3 \\ -7 \\ 5 \end{bmatrix} , \vec{x} = \begin{bmatrix} 4 \\ 10 \\ -7 \end{bmatrix} .$$

Explain why the set $\beta = \{\vec{b}_1,\ \vec{b}_2\}$ can be considered as a basis for a subspace H. Geometrically describe H. Is $\vec{x}$ in H? If so, give the coordinates of $\vec{x}$ relative to the basis β.

Solution: Any linearly independent set of vectors (all with the same number of entries) can be used as a basis for a subspace that is the span of those vectors. Since $\vec{b}_1$ and $\vec{b}_2$ are linearly independent (one is not a constant multiple of the other), they serve as a basis for the subspace H=spanβ. Since there are two basis vectors, H is two dimensional, and is therefore a plane, which is a subspace of $\mathbb{R}^3$, since the basis vectors each have 3 entries.

To find if $\vec{x}$ is in H, we need to see if $\vec{x}$ can be written as a linear combination of $\vec{b}_1$ and $\vec{b}_2$. The easiest way to do this is to make the augmented matrix

$$A = \begin{bmatrix} 1 & -3 & 4 \\ 5 & -7 & 10 \\ -3 & 5 & -7 \end{bmatrix}$$

and row reduce. First replace row 2 with itself minus 5 times row 1, then replace row 3 with itself plus 3 times row 1, then replace row 3 with itself plus one half of row 2 (now in echelon form), scale row 2 by dividing by 8, then finally replace row 1 with itself plus 3 times row 2, leaving

$$A = \begin{bmatrix} 1 & 0 & 1/4 \\ 0 & 1 & -5/4 \\ 0 & 0 & 0 \end{bmatrix}$$

There is no pivot in the augmented column, so the system is consistent, meaning that $\vec{x}$ is in fact a linear combination of the vectors in β. From our reduced echelon form, we find that the weights needed for $\vec{b}_1$ and $\vec{b}_2$ to obtain $\vec{x}$ as a linear combination are 1/4 and -5/4, respectively. So, the coordinates of $\vec{x}$ relative to the basis β are (1/4, -5/4).

3. Answer the following short questions, justifying your answers fully:

   (a) If M is a 3×5 matrix, and its column space is $\mathbb{R}^3$, does that mean its null space is $\mathbb{R}^2$? If so, explain why, if not, explain what the null space of M actually is. **Solution**: Each column of M is in $\mathbb{R}^3$, and since the column space of M is all of $\mathbb{R}^3$, then there are three pivot columns of M. Since there are 5 columns, though, that means there are 2 free variables, hence the null space is two dimensional. But, that is not equivalent to saying that the null space is $\mathbb{R}^2$ - vectors in $\mathbb{R}^2$ have 2 components, while vectors in the null space have 5 components (since M has 5 columns), so they are in $\mathbb{R}^5$. Hence, the null space of M is not $\mathbb{R}^2$, but is rather a planar (two-dimensional) subspace of $\mathbb{R}^5$.

(b) Suppose β is a set of vectors that is a basis for a subspace H. If I create a new set of vectors α that includes all of the vectors in β, but also includes one more vector that is a linear combination of some of the vectors of β, is the span of the set α equal to H? **Solution**: Yes. The span of the linearly independent vectors in β is H, since they are a basis for H. Creating a new vector $\vec{u}$ that is a linear combination of some of the vectors in β just gives you a vector that is already in H. Adding such a vector to the set β to create a new set α does not increase the size of the subspace that is spanned, as every linear combination of the basis vectors (of which $\vec{u}$ is an example) is already within the subspace.

(c) Is it possible for the null space of an $m \times n$ matrix to be $\mathbb{R}^n$? If so, under what circumstances? **Solution**: The only way this could happen is if the null space of the matrix had n basis vectors. This would mean the orignal matrix had n free variables, meaning it would have 0 pivot columns. This is only possible for the zero matrix. That is, the only way that $A\vec{x} = \vec{0}$ has a solution for *all* $\vec{x}$ in $\mathbb{R}^n$ is if all entries of A are zero.