Worksheet 7

1. Compute the determinant of

$$A = \begin{bmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 3 & 0 & 0 & 0 \\ 8 & 3 & 1 & 7 \end{bmatrix}$$

using a cofactor expansion. Choose whatever expansion leads to the least amount of work.

Solution: The best expansion to choose first is the one along row 3, as only one entry in row 3 is nonzero. Hence, the determinant we are after is given by $3C_{3,1}$, where $C_{3,1}$ is the determinant of the submatrix

$$A_{3,1} = \begin{bmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 7 \end{bmatrix}$$

This determinant can be found by using a cofactor expansion along the first row, which again only has one nonzero entry. The determinant of this submatrix is therefore $5C_{1,3}$, where $C_{1,3}$ is the determinant of the submatrix

$$A_{1,3} = \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix} .$$

The determinant of $A_{1,3}$ is easily found to be $7 \cdot 1 - 2 \cdot 3 = 1$. Hence, the determinant of $A_{3,1} = 5 \cdot 1 = 5$, hence the determinant of A is $3 \cdot 5 = 15$.

2. Explain why, if two rows of the matrix A are equal, then the determinant of A is zero.

Solution: If two rows of the matrix A are equal, then two columns of A^T are equal. But, then the columns of A^T are not linearly independent, so A^T is not invertible, so the determinant of A^T is zero. But, the determinant of A is equal to the determinant of A^T, so the determinant of A is zero also.

3. Find the determinant of

$$A = \begin{bmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{bmatrix}$$

by row reduction to echelon form.

Solution: If we row reduce to echelon form, then we need only take the product of the elements along the diagonal to find the determinant of the reduced matrix. If we keep track of row interchanges and scalings along the way, we can find the determinant of our original matrix. To reduce to echelon form, replace row 2 with itself plus 2 times row 1, replace row 3 with itself minus 3 times row 1, replace row 4 with itself minus row 1, replace row 3 with itself plus 4 times row 2, then replace row 4 with itself plus 4 times row 2, then finally replace row 4 with itself minus row 3, leaving

$$A = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & 30 & 27 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We made no scalings or interchanges, so the determinant of this reduced matrix is equal to the determinant of our original matrix. The determinant of the reduced matrix is $1 \cdot 1 \cdot 30 \cdot 0 = 0$, so the determinant of A is zero.